

Lecture 1

1) Introduction:

The understanding of many physical phenomena depend on statistical and probabilistic concepts:

Statistical Mechanics (physics of systems composed of many parts: gases, liquids, solids.)

1 mole of anything contains 6×10^{23} particles (Avogadro's number)

impossible to keep track of all 6×10^{23} particles even with the fastest computer imaginable

resort to learning about the group properties of all the particles

partition function: from partition function can calculate energy, entropy, pressure... of a system

Quantum Mechanics (physics at the atomic or smaller scale)

wavefunction = probability amplitude

talk about the probability of an electron being located at (x,y,z) at a certain time.

The understanding and interpretation of all experimental data depend on statistical and probabilistic concepts:

how do we extract the best value of a quantity from a set of measurements?

how do we decide if our experiment is consistent/inconsistent with a given theory?

how do we decide if our experiment is internally consistent?

how do we decide if our experiment is consistent with other experiments?

In this course we will concentrate on the above experimental issues!

2) Brief Discussion of Probability

a) *Definition of probability:*

Lets define probability by example: Suppose we have N trials and a specified event occurs r times.

For example the trial could be rolling a dice and the event could be a six coming up.

We define the probability (P) of an event (E) occurring as:

$$P(E) = r/N \text{ when } N \rightarrow \infty$$

Examples:

coin toss $P(\text{heads}) = 0.5$

six sided dice $P(1) = 1/6$

Remember: $P(\text{heads})$ should approach 0.5 the more times you toss the coin. Obviously for a single coin toss we can never get $P(\text{heads}) = 0.5!$

By definition probability is a non-negative number bounded by

$$0 \leq P \leq 1$$

If $P = 0$ then the event never occurs

If $P = 1$ then the event always occurs

The sum (or integrate) of all probabilities if they are mutually exclusive should equal to 1.

b) Probability can be a discrete or continuous variable:

Discrete case:

example of discrete case: tossing a six-sided dice.

$$P(x_i) = P_i \quad \text{here } x_i = 1, 2, 3, 4, 5, 6 \text{ and } P_i = 1/6 \text{ for all } x_i.$$

In the discrete case only certain values of P are allowed.

Another discrete example is tossing a coin. Only 2 choices, heads or tails.

NOTATION: x_i is call a "random variable"

For both of the above discrete cases (and in general) when we sum over all possibilities:

$$\sum_i P(x_i) = 1$$

Continuous Probability:

In this case P can be any number between 0 and 1.

We can define a "probability density function" $f(x)$

$$f(x)dx = P(x \leq \alpha \leq x + dx) \quad \alpha \text{ is a continuous variable}$$

Just like the discrete case the sum of all probabilities must equal 1.

For the continuous case this means:

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

NOTE: The probability for x to be exactly some number is zero since:

$$\int_{x=a}^{x=a} f(x)dx = 0$$

Aside: Probability theory is an interesting branch of mathematics.

Calculus of Probabilities \equiv set theory.

Examples of $P(x)$ and $f(x)$:

<u>Discrete = $P(x)$</u>	<u>Continuous = $f(x)$</u>
binomial	uniform, i.e. $f(x) = \text{constant}$
Poisson	Gaussian
	exponential
	chi square

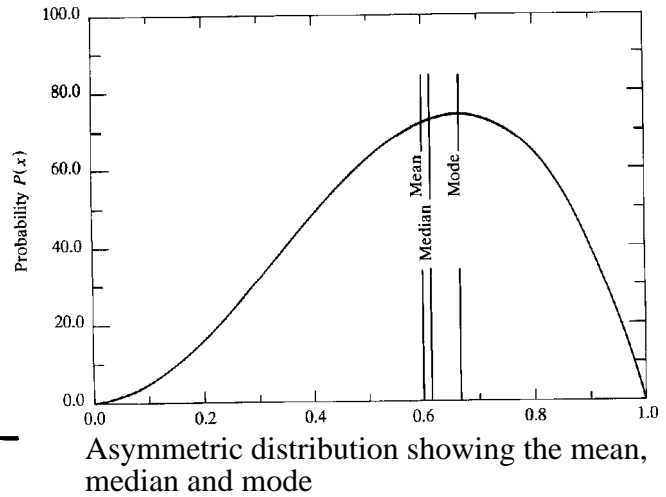
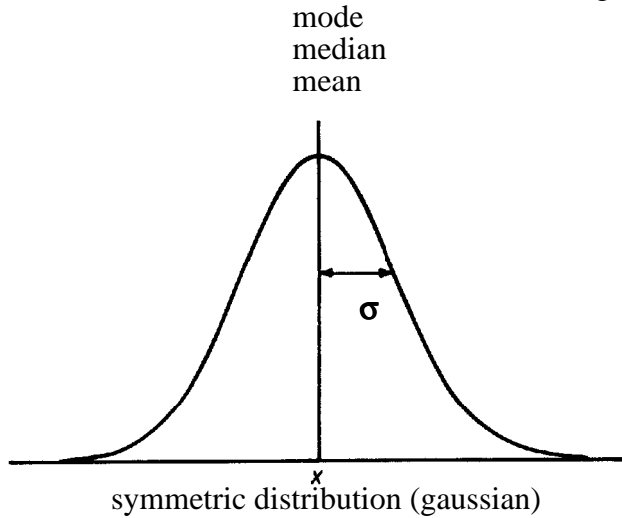
c) How do we classify distributions?

Consider a continuous probability distribution $f(x)$. The following quantities are very useful:

Mean	Mode	Median	Variance
average	most probable	50% point	width of distribution
$\mu = \int_{-\infty}^{+\infty} xf(x)dx$	$\left. \frac{\partial f(x)}{\partial x} \right _{x=a} = 0$	$0.5 = \int_{-\infty}^a f(x)dx$	$\sigma^2 = \int_{-\infty}^{+\infty} f(x)(x - \mu)^2 dx$

For some $f(x)$'s (e.g. Gaussian distribution) the mean, median, and mode are all at the same x .

However, this is not true for most distributions (see figure below, right).



3) Calculation of mean and variance:

As an example consider a discrete data set consisting of three numbers: {1, 2, 3}

The average (μ) is just:

$$\mu = \sum_{i=1}^n x_i / n = \frac{1+2+3}{3} = 2$$

Complication: Suppose some measurement are more precise than others. Our averaging procedure should reflect this. Let each measurement x_i have a weight w_i associated with it. The average is now:

$$\mu = \sum_{i=1}^n x_i w_i / \sum_{i=1}^n w_i$$

The variance (σ^2 , σ is called the standard deviation), or the average squared deviation from the mean is just:

$$\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 / n$$

We can rewrite the above expression by expanding the summations:

$$\sigma^2 = \frac{1}{n} \left[\sum_{i=1}^n x_i^2 + \sum_{i=1}^n \mu^2 - 2\mu \sum_{i=1}^n x_i \right]$$

We can simplify the above using:

$$\frac{1}{n} \sum_{i=1}^n \mu^2 = \mu^2 \quad \text{and} \quad \frac{2\mu}{n} \sum_{i=1}^n x_i = 2\mu^2$$

Finally we wind up with:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 + \mu^2 - 2\mu^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \mu^2$$

Note: the n in the denominator would be $n - 1$ if we determined the average (μ) from the data itself. Using the definition of μ from above we have for our example of $\{1,2,3\}$:

$$\sigma^2 = \sum_{i=1}^n x_i^2 / n - \left(\sum_{i=1}^n x_i / n \right)^2 = 4.67 - 2^2 = 0.67$$

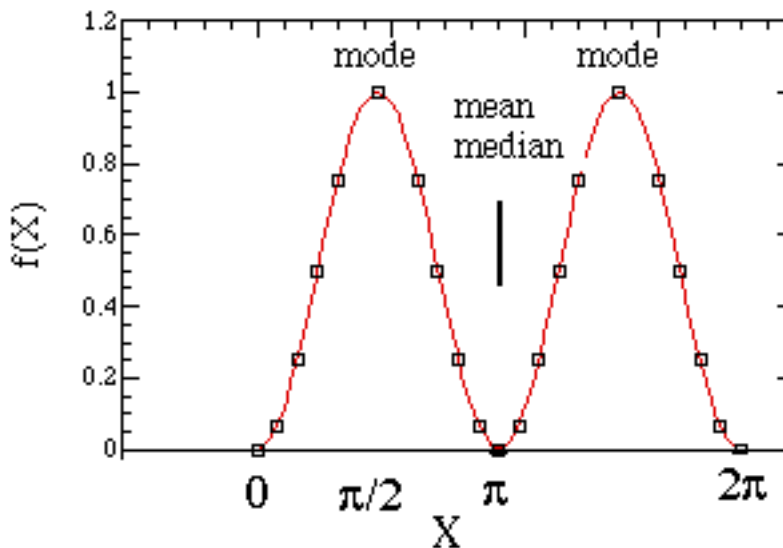
Again, the case where the measurements have different weights is more complicated:

$$\sigma^2 = \sum_{i=1}^n w_i (x_i - \mu)^2 / \sum_{i=1}^n w_i = \sum_{i=1}^n w_i x_i^2 / \sum_{i=1}^n w_i - \mu^2$$

In the above formula μ is the *weighted* mean. Finally if we calculated μ from the data the above gets multiplied by a factor $n / (n - 1)$.

Let's now consider the case of continuous probability distribution:

Let $f(x) = \sin^2 x$ for $0 \leq x \leq 2\pi$



This function has 2 modes! Also the mean and median are at the same place, but differ from the mode(s). For continuous probability distributions the mean, mode, and median are calculated using either integrals or derivatives, as shown below.

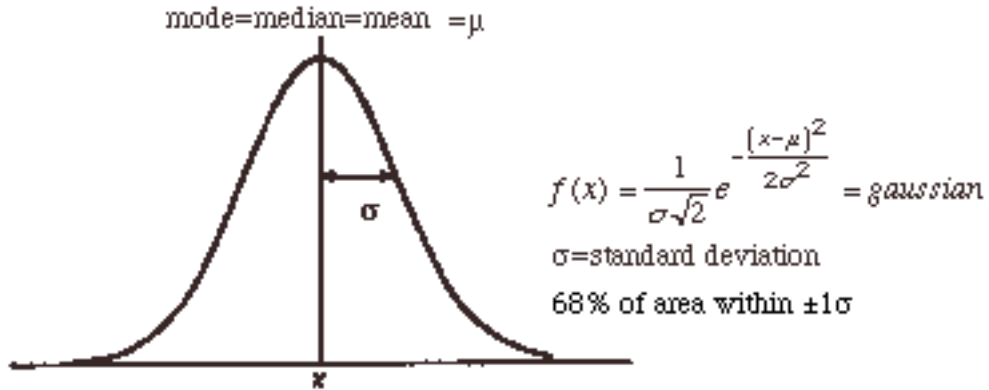
In this class we rely on tables of integrals and derivatives to avoid getting bogged down in the details of the calculus.

$$\mu = \int_0^{2\pi} x \sin^2 x dx / \int_0^{2\pi} \sin^2 x dx = \pi$$

$$mode = \frac{\partial}{\partial x} (\sin^2 x) = 0 \Rightarrow \frac{\pi}{2}, \frac{3\pi}{2}$$

$$median = \int_0^{\alpha} \sin^2 x dx / \int_0^{2\pi} \sin^2 x dx = \frac{1}{2} \Rightarrow \alpha = \pi$$

One more example of a continuous probability distribution: the Gaussian distribution function.

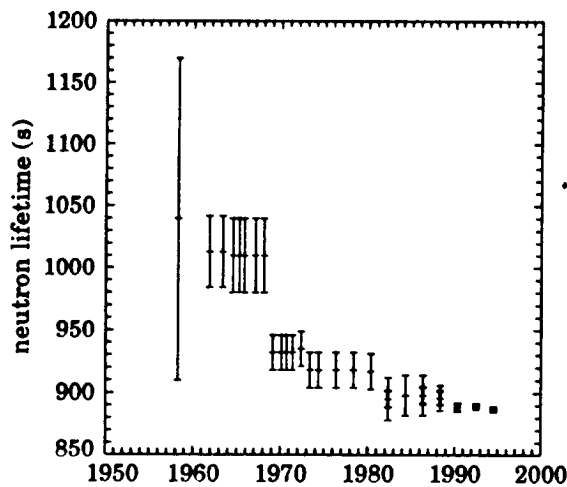


3) Some comments on accuracy, precision, and errors:

a) Accuracy: The accuracy of an experiment refers to how close the experimental measurement is to the true value of the quantity being measured.

b) Precision: This refers to how well the experimental result has been determined, without regard to the true value of the quantity being measured.

Note: Just because an experiment is precise does not mean it is accurate!?!



The figure above shows various measurements of the neutron lifetime over the years. Note the big jump downward in the early 1960's. Are any of these measurements accurate?

Very often we use the relative precision as an indicator of how "good" an experiment is.

Relative precision is defined by the following example:

Suppose we measure the length of a table to be 10 inches with an uncertainty of 1 inch. The relative precision is:

$$\text{Relative Precision} = \text{uncertainty}/\text{value} = 1/10 = 0.1$$

The uncertainty is usually given by the standard deviation (lots more about that later).

Sometimes we express relative precision as a percentage. For the above example we would have:

$$\% \text{ Relative Precision} = 100*(1/10) = 10\%$$

c) Results from experiments are often presented as:

$$N \pm XX \pm YY$$

N : value of quantity measured (or determined) by experiment.

XX : statistical error, usually assumed to be from a Gaussian distribution.

With the assumption of Gaussian statistics we can say (calculate) something about how well our experiment agrees with other experiments and/or theories.

Expect an 68% chance that the true value is between $N - XX$ and $N + XX$.

YY : systematic error. Hard to estimate, distribution of errors usually not known.

Examples: mass of proton = 0.9382769 ± 0.0000027 GeV

mass of W boson = $80.8 \pm 1.5 \pm 2.4$ GeV

What's the difference between statistical and systematic errors?

statistical errors are "random" in the sense that if we repeat the measurement enough times:

$$XX \rightarrow 0$$

However systematic errors *do not* $\rightarrow 0$ with repetition.

Some examples of sources of systematic errors:

voltmeter not calibrated properly

a ruler not the length we think is (meter stick might really be $<$ meter!)

Note: Because of systematic errors one can have an experimental result that is precise, but not accurate!

How do we combine systematic and statistical errors to get one estimate of precision?

BIG PROBLEM!

Two choices:

$$\sigma_{\text{tot}} = XX + YY \text{ add them linearly}$$

$$\sigma_{\text{tot}} = (XX^2 + YY^2)^{1/2} \text{ add them in quadrature}$$

Some other ways of quoting experimental results

lower limit: "the mass of particle X is > 100 GeV"

upper limit: "the mass of particle X is < 100 GeV"

asymmetric errors: mass of particle $X = 100_{-3}^{+4}$ GeV

