

Maximum LogLikelihood Fit- Radioactive Decay

PDF for radioactive decay: $F(t) = Ne^{-\frac{t}{\tau}}$

Normalize in range 0-10 μsec $\int_0^{10} Ne^{-\frac{t}{\tau}} dt = \tau(1 - e^{-\frac{10}{\tau}}) \Rightarrow N = \frac{1}{\tau(1 - e^{-\frac{10}{\tau}})}$

The Likelihood function is: $L = \prod_i F(t_i) = \prod_i \frac{e^{-\frac{t_i}{\tau}}}{\tau(1 - e^{-\frac{10}{\tau}})}$

Thus we want to maximize the logLikelihood:

$$\ln(L) = \sum_i \ln\left(\frac{e^{-\frac{t_i}{\tau}}}{\tau(1 - e^{-\frac{10}{\tau}})}\right) = \frac{-\sum_i t_i}{\tau} - N \ln(\tau) - N \ln\left(1 - e^{-\frac{10}{\tau}}\right) \quad \text{Eqn.1}$$

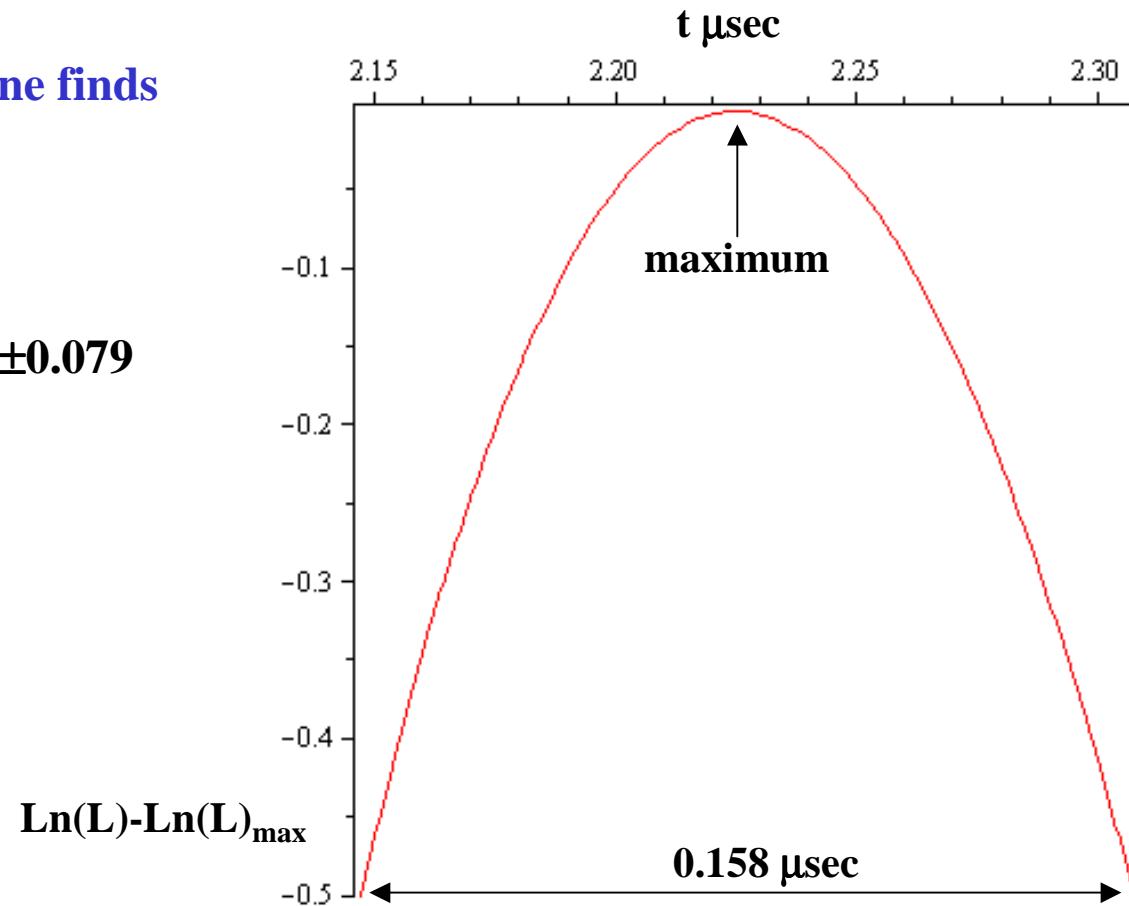
Generate 1000 MC events in Range $0 < t < 10 \mu\text{sec}$ with Lifetime $2.2 \mu\text{sec}$

$$N = 1000 \quad S \equiv \sum_i t_i = 2112.085$$

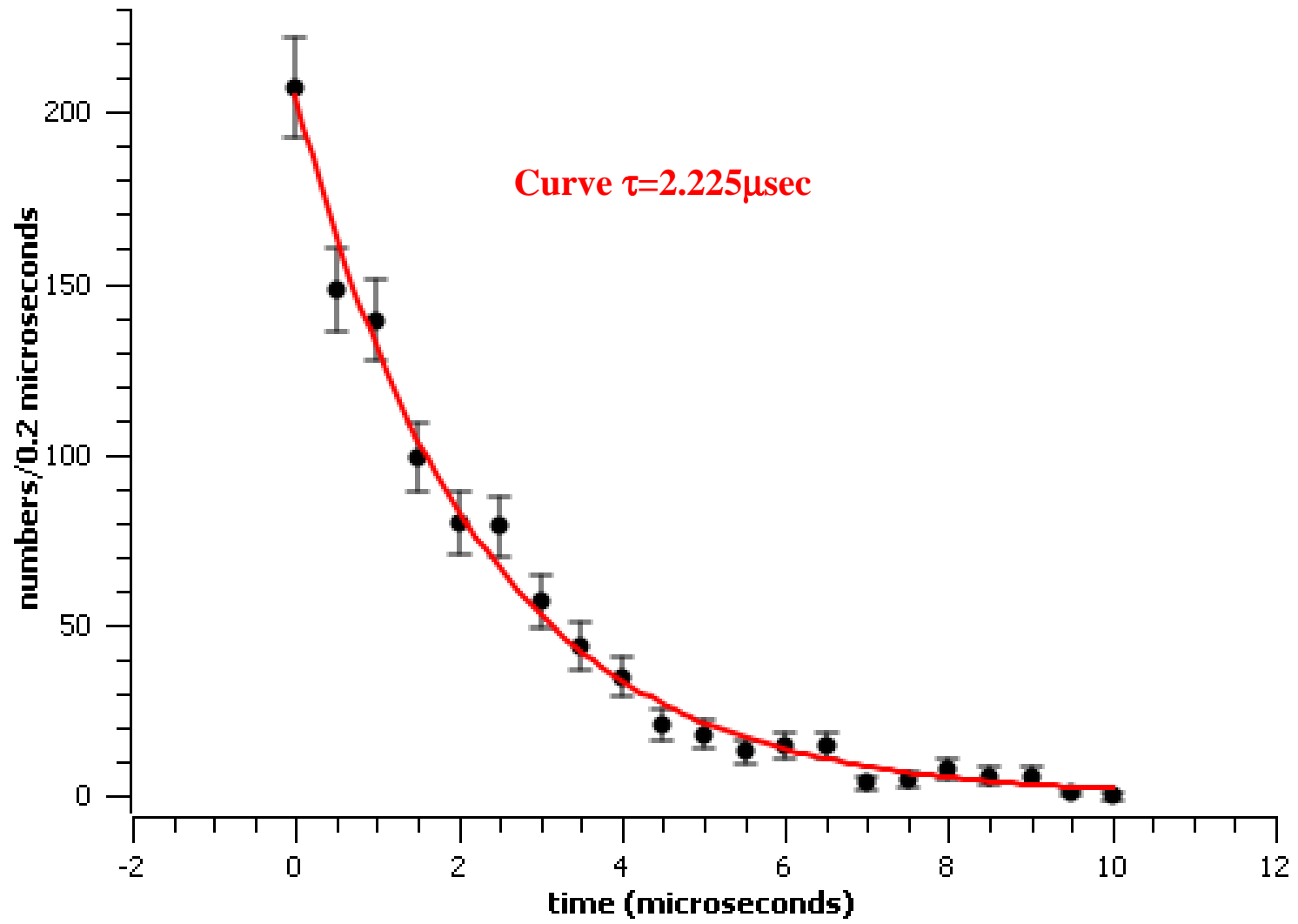
Maximize Eqn 1: $\tau = 2.225 \mu\text{sec}$ $\ln(L)_{\text{max}} = -1737.8$

When $\ln(L) - \ln(L)_{\text{max}} = -0.5$ one finds the 1σ errors

Thus our fit yields: $\tau = 2.225 \pm 0.079$



Muon Lifetime Log-Likelihood Fit (1000 samples)



Alternate Calculation of σ_τ

Mean Sigma Parent Distribution

$$\int_0^{10} t \frac{e^{-\frac{t}{2.2}}}{2.2(1 - e^{-\frac{10}{2.2}})} dt = 2.0927 \quad \int_0^{10} (t - 2.0927)^2 \frac{e^{-\frac{t}{2.2}}}{2.2(1 - e^{-\frac{10}{2.2}})} dt = 3.756$$

Differential of Eqn 1, set to zero to get maximum: $\frac{S}{N} = \tau - \frac{10e^{-\frac{10}{\tau}}}{(1 - e^{-\frac{10}{\tau}})}$

Propagate errors: $\sigma_{\frac{S}{N}}^2 = \left(\frac{d(\frac{S}{N})}{d\tau} \right)^2 \sigma_\tau^2 = \left(1 - \frac{100e^{-\frac{10}{\tau}}}{\tau^2(1 - e^{-\frac{10}{\tau}})} - \frac{100e^{-\frac{20}{\tau}}}{\tau^2(1 - e^{-\frac{10}{\tau}})^2} \right)^2 \sigma_\tau^2$

Plugging in Numbers $\sigma_{\frac{S}{N}}^2 = 3.765$ $\tau = 2.2$ yields $\sigma_\tau^2 = 6.238$

Using Central Limit Theorem $\sigma_{fit} = \frac{\sqrt{6.238}}{\sqrt{1000}} = 0.079$

To test Baye's Theorem we run the Monte Carlo simulation 1 million times and histogram the best fit τ . The resulting plot yields a mean of $2.2 \mu\text{sec}$ with an RMS of $0.079 \mu\text{sec}$. This agrees with our previous single run error bars. Thus we have made a correct determination of the uncertainty of our data. (e.g. 67% of the time our answer will be within $\pm 0.079 \mu\text{sec}$ of $2.225 \mu\text{sec}$)

