An ideal gas consists of \( N \) massless identical bosons (with energy-momentum relation \( \varepsilon = pc \)), all in the same spin state, in a volume \( V \) at temperature \( T \).

(a) For \( T > T_c \), express \( N \) as an integral that depends on \( T \) and the chemical potential \( \mu \).
\[
N = \frac{V}{\hbar^3} \int d^3p \frac{1}{e^{\beta (\varepsilon - \mu)} - 1} = \frac{V}{\hbar^3} \int_0^\infty d\varepsilon \int_0^\infty d\rho \rho^2 \frac{1}{e^{\beta (\rho c - \mu)} - 1}
\]
\[
= \frac{4\pi V}{\hbar^2 c^2} \int_0^\infty d\varepsilon \frac{\varepsilon^2}{e^{\beta \varepsilon} - 1}
\]

(b) For \( T < T_c \), express \( N \) in terms of the condensate number \( N_0 \) and an integral that depends on \( T \).
\[
N = N_0 + \frac{V}{\hbar^3} \int d^3p \frac{1}{e^{\beta \varepsilon} - 1}
\]
\[
= N_0 + \frac{4\pi V}{\hbar^2 c^2} \int_0^\infty d\varepsilon \frac{\varepsilon^2}{e^{\beta \varepsilon} - 1}
\]

(c) At the phase transition for Bose-Einstein condensation, \( T = T_c, \mu = 0 \) and \( N_0 = 0 \). Solve for the critical temperature \( T_c \) as a function of \( N/V \).
\[
N = \frac{4\pi V}{\hbar^2 c^2} \int_0^\infty d\varepsilon \frac{\varepsilon^2}{e^{\beta \varepsilon} - 1} = \frac{4\pi V}{\hbar^2 c^2} \int_0^\infty dx \frac{x^2}{e^{\frac{x}{\kappa}} - 1}
\]
\[
= \frac{4\pi V}{\hbar^2 c^2} (kT_c)^3 \cdot 2 \zeta(3)
\]
\[
kT_c = \hbar c \left( \frac{1}{8 - \pi \zeta(3)} \frac{N}{V} \right)^{\frac{1}{3}}
\]

integral table:
\[
\int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6}, \quad \int_0^\infty dx \frac{x^2}{e^x - 1} = 2\zeta(3), \quad \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}.
\]