Goldenfeld, Chapter 9

Problem 9.4

The Hamiltonian can be separated into terms with that depend only on spins in the same block and terms that depend on spins in two blocks:

$$\mathcal{H} = \mathcal{H}_0 + V$$

$$\mathcal{H}_0 = \sum_I \left(K \sum_{i,j \in I} S_i S_j + h \sum_{i \in I} S_i \right)$$

$$V = \sum_{\langle IJ \rangle} \left(K \sum_{\substack{\langle i,j \rangle \\ i \in I \\ j \in J}} S_i S_j \right)$$

We will treat V as a perturbation.

The partition function Z_0 associated with $\mathcal{H}_0$ factors into partition functions Z_I for each of the blocks. The sum of the Boltzmann factors over the spin configurations σ_I defined in Eq (9.147) are

$$S_I = +1 : \quad 3 e^{-K+h} + e^{3K+3h}$$

$$S_I = -1 : \quad 3 e^{-K-h} + e^{+3K-3h}$$

This can be expressed as

$$3 e^{-K+hS_I} + e^{3K+3hS_I} = e^{A+BS_I}$$

The cases $S_I = +1$ and -1 provide two equations for the two unknowns A, B:

$$3e^{-K+h} + e^{3K+3h} = e^{A+B}$$

$$3e^{-K-h} + e^{3K-3h} = e^{A-B}$$

$$e^{2A} = \left(3e^{-K+h} + e^{3K+3h}\right)\left(3e^{-K-h} + e^{3K-3h}\right)$$

$$= \left(e^{3K} + 3e^{-K-2h}\right)\left(e^{3K} + 3e^{-K+2h}\right)$$

$$e^{2B} = \left(3e^{-K+h} + e^{3K+3h}\right) / \left(3e^{-K+h} + e^{3K-3h}\right)$$

$$= e^{6h}\left(e^{3K} + 3e^{-K-2h}\right) / \left(e^{3K} + 3e^{-K+2h}\right)$$

The contribution to the partition function from V can be expressed as a multiplicative factor. The leading contribution in the cumulant expansion is $e^{\langle V\rangle_0}$. The expectation value of the term in V associated with the nearest neighbor blocks I and J can be expressed as in Eq. (9.169):

$$\langle V_{IJ}\rangle_0 = 2K \langle S_3^J\rangle_0 \langle S_1^I\rangle_0$$

The expectation value of S_3^J depends on the block spin S^J:

$$S_J=+1: \quad \langle S_3^J\rangle = \left(e^{-K+h}+e^{3K+3h}\right)/\left(3e^{-K+h}+e^{3K+3h}\right)$$

$$= \Phi(K+\tfrac{1}{2}h)$$

$$S_J=-1: \quad \langle S_3^J\rangle = \left(-e^{-K-h}-e^{3K-3h}\right)/\left(3e^{-K-h}+e^{3K-3h}\right)$$

$$= -\Phi(K-\tfrac{1}{2}h)$$

$$\text{where } \Phi(x) = \frac{1+e^{-4x}}{1+3e^{-4x}}$$

This can be expressed as

$$\frac{1+S_J}{2}\Phi(K+\tfrac{1}{2}h) + \frac{1-S_J}{2}\left[-\Phi(K-\tfrac{1}{2}h)\right]$$

$$= \tfrac{1}{2}\left[\Phi(K+\tfrac{1}{2}h)-\Phi(K-\tfrac{1}{2}h)\right] + \tfrac{1}{2}\left[\Phi(K+\tfrac{1}{2}h)+\Phi(K-\tfrac{1}{2}h)\right]S_J$$

The expectation value of V_{IJ} is then

$$\langle V_{IJ}\rangle_0 = \tfrac{1}{2}K\left[\Phi(K+\tfrac{1}{2}h)-\Phi(K-\tfrac{1}{2}h)\right]^2$$

$$+K\left[\Phi^2(K+\tfrac{1}{2}h)-\Phi^2(K-\tfrac{1}{2}h)\right](S_I+S_J)$$

$$+\tfrac{1}{2}K\left[\Phi(K+\tfrac{1}{2}h)+\Phi(K-\tfrac{1}{2}h)\right]^2 S_I S_J$$

We can now deduce the form of the effective Hamiltonian for block spins:

$$\mathcal{H}' = M \cdot F_0 + K' \sum_{\langle IJ \rangle} S_I S_J + h' \sum_I S_I$$

$$K' = \tfrac{1}{2} K \left[\Phi(K + \tfrac{1}{2}h) + \Phi(K - \tfrac{1}{2}h) \right]^2$$

$$h' = 3h + \tfrac{1}{2} \log \frac{1 + 3e^{-4K-2h}}{1 + 3e^{-4K+2h}} + 3K \left[\Phi^2(K + \tfrac{1}{2}h) - \Phi^2(K - \tfrac{1}{2}h) \right]$$

I have not plotted the RG flow diagram, but I assume that it resembles Figure 9.7 of Goldenfeld.

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Exercise 9-5

The results from Exercise 3-2 on the two largest eigenvalues of the transfer matrix on an $M \times N$ lattice in the limit $N \to \infty$ are

$M=1$: $\lambda_1 = 1 + x^2$, $\lambda_2 = x^2 - 1$

$M=2$: $\lambda_1 = \frac{1}{2}\left(x^4 + 2 + x^{-4} + \sqrt{x^8 + 14 + x^{-8}}\right)$, $\lambda_2 = x^4 - 1$

where $x = e^K$. The correlation length ξ is given by

$$\xi^{-1} = \log \frac{\lambda_1}{\lambda_2}$$

According to the finite-size scaling relation in Eq. (9.261), the behavior of the correlation length in the limit $t \to 0$ is

$$\frac{L}{\xi_L} = A + B t L^{1/\nu}$$

where L^d is the size of the system. The derivation of this relation is equally valid if L is the finite length of one dimension and the other dimensions have infinite lengths. Thus we have

$$\frac{M}{\xi_M} = A + B t M^{1/\nu}$$

At the critical point $K = K_*$ or $t = 0$, this reduces to

$$\left.\frac{M}{\xi_M}\right|_{t=0} = A$$

Setting the left side equal for $M=1$ and $M=2$, we get

$$\log \frac{1+x^2}{x^2-1} = 2 \log \frac{x^4+2+x^{-4}+\sqrt{x^8+14+x^{-8}}}{2(x^4-1)}$$

The solution is $x_* = 1.546$, which implies

$$K_* = 0.4357$$

This is close to the analytic result of Onsager:

$$K_* = \tfrac{1}{2}\log(1+\sqrt{2}) = 0.4407$$

By differentiating the finite size scaling relation with respect to t, we get

$$\left.\frac{\partial}{\partial t}\frac{M}{\xi_M}\right|_{t=0} = M^{1/\nu}$$

Dividing this equation for $M=2$ by the corresponding equation for $M=1$, we get

$$\left.\frac{\frac{\partial}{\partial t}\frac{2}{\xi_2}}{\frac{\partial}{\partial t}\frac{1}{\xi_1}}\right|_{t=0} = 2^{1/\nu}$$

The derivative can be replaced by derivative with respect to K or by derivative with respect to $x = e^K$.

$$\left. \frac{2 \frac{d}{dx} \log \frac{x^4 + 2 + x^{-4} + \sqrt{x^8 + 14 + x^{-8}}}{2(x^4 - 1)}}{\frac{d}{dx} \log \frac{1 + x^2}{x^2 - 1}} \right|_{x = x_*} = 2^{1/\nu}$$

The result for ν is

$$\nu = 0.9873$$

This is close to the analytic result $\nu = 1$.