Goldenfeld, Chapter 9

Exercise 9-1

(a) Kadanoff's scaling arguments give the results for the critical exponents in Eqs. (9.34–36):

$$\nu = \frac{1}{y_t}$$

$$\eta = d+2-2y_h$$

$$\Delta = \frac{y_h}{y_t}$$

Scaling argument can also be used to express the critical exponent δ in terms of y_h and y_t. The scaling relation for the most singular term in the free energy density is given in Eq. (9.17).

$$f_s(t,h) = l^{-d} f_s(l^{y_t} t, l^{y_h} h)$$

Differentiating both sides with respect to t, we get

$$\frac{\partial f_s}{\partial h}(t,h) = l^{-d} \frac{\partial f_s}{\partial h}\left(l^{y_t} t, l^{y_h} h\right) l^{y_h}$$

If we choose l so $l^{y_h} h = 1$, this becomes

$$\frac{\partial f_s}{\partial h}(t,h) = \left(\frac{1}{h}\right)^{(y_h-d)/y_h} \frac{\partial f_s}{\partial h}\left(\left(\frac{1}{h}\right)^{y_t/y_h} t, 1\right)$$

Setting $t=0$, this reduces to

$$\frac{\partial f_s}{\partial h}(0,h) = h^{(d-y_h)/y_h}\frac{\partial f_s}{\partial h}(0,1)$$

The left side is proportional to the magnetization at the critical temperature, so this implies

$$m \sim h^{(d-y_h)/y_h}$$

We can read off the critical exponent δ:

$$\delta = \frac{y_h}{d-y_h}$$

The exponent y_h can be expressed in terms of η:

$$y_h = \frac{d+2-\eta}{2}$$

Inserting this into the expression for δ, we find

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

(b) If we chose l so $l^{y_t}|t|=1$ in the scaling equation for the free energy density, it becomes

$$f_s(t,h) = \left(\frac{1}{|t|}\right)^{-d/y_t} f_s\left(\pm 1, \left(\frac{1}{|t|}\right)^{y_h/y_t} h\right)$$

where the first argument on the right side is the sign of t. We consider the possibility that the exponent y_t can be different for $t>0$ and $t<0$. Since the critical exponent for the correlation length is $\nu = 1/y_t$, our scaling relation can be written

$$f_s(t,h) = |t|^{d\nu} f_s\left(\pm 1, |t|^{-y_h \nu} h\right)$$

We can also express this as a power of h multiplied by a function of $|t|^{-y_h\nu} h$:

$$f_s(t,h) = h^{+d/y_h}\left(\frac{h}{|t|^{y_h\nu}}\right)^{-d/y_h} f_s\left(\pm 1, h/|t|^{y_h\nu}\right)$$

$$= h^{d/y_h}\, \phi_\pm\left(h/|t|^{y_h\nu}\right)$$

We consider the possibility that the correlation length exponent could have a value ν' for $t<0$ that is different from its value ν for $t>0$.

For $h>0$, the free energy $f_s(t,h)$ must be a smooth function of t as $t\to 0$. Continuity of $f_s(t,h)$ as $t \to 0$ requires

$$\phi_+(\infty) = \phi_-(\infty).$$

Smoothness also requires the derivatives with respect to t and h to be continuous. The logarithmic derivatives are

$$t \frac{\partial}{\partial t} f_s(t,h) = h^{d/y_h} \phi_\pm'\left(h/|t|^{y_h\nu}\right)\left(-y_h\nu \frac{h}{|t|^{y_h\nu}}\right)\left(\pm\frac{t}{|t|}\right)$$

$$h \frac{\partial}{\partial h} f_s(t,h) = \frac{d}{y_h} h^{d/y_h} \phi_\pm\left(h/|t|^{y_h\nu}\right) + h^{d/y_h} \phi_\pm'\left(h/|t|^{y_h\nu}\right)\frac{h}{|t|^{y_h\nu}}$$

The conditions of smoothness at $t=0$ can be written

$$-y_h\nu\left[x\,\phi_+'(x)\right]_{x=\infty} = -y_h\nu'\left[x\,\phi_+'(x)\right]_{x=\infty}$$

$$\frac{d}{y_h}\phi_+(\infty) + \left[x\,\phi_+'(x)\right]_{x=\infty} = \frac{d}{y_h}\phi_-(\infty) + \left[x\,\phi_-'(x)\right]_{x=\infty}$$

The 3 conditions together require $\nu' = \nu$.

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Exercise 9-2

(a) Among the evidence that "mean field theory works above $d=4$" is that the Ginsburg criterion is satisfied in the limit $t \to 0$ for $d > 4$.

"Mean field theory violates hyperscaling because $\alpha = 0$ and $\nu = \frac{1}{2}$". The hyperscaling law that is violated is the Josephson scaling law in Eq. (8.3)

$$2 - \alpha = \nu d$$

This is consistent with $\alpha = 0$ and $\nu = \frac{1}{2}$ only if $d = 4$.

(b) The singular part of the free energy density satisfies the scaling relation

$$f_s(t, h, \tilde{K}_3) = t^{d/y_t} f_s(1, h t^{-y_h/y_t}, \tilde{K}_3 t^{-y_3/y_t})$$

If f_s is smooth as $\tilde{K}_3 \to 0$, we have

$$f_s(t, h, 0) = t^{d/y_t} f_s(1, h t^{-y_h/y_t}, 0)$$

which implies the usual scaling relations.

Suppose the behavior as $\tilde{K}_3 \to 0$ is singular:

$$f_s(t, h, \tilde{K}_3) \longrightarrow \tilde{K}_3^{-\mu} \bar{f}(t,h)$$

The limiting behavior of both side of the scaling relation is

$$(\tilde{K}_3)^{-\mu} \bar{f}(t,h) = t^{d/y_t} (\tilde{K}_3 t^{-y_3/y_t})^{-\mu} \bar{f}(1, h t^{-y_h/y_t})$$

This implies

$$\bar{f}(t,h) = t^{(d+\mu y_3)/y_t} \bar{f}(1, h t^{-y_h/y_t})$$

This disagrees with the usual scaling relation.

The analog of the Josephson hyperscaling law is obtained by differentiating twice with respect to t:

$$C \sim \left(\frac{\partial}{\partial t}\right)^2 f_s(t,h)$$

$$t^{-\alpha} \sim t^{(d+\mu y_3)/y_t - 2}$$

$$\alpha = 2 - \frac{d+\mu y_3}{y_t}$$

Combining the with $\nu = 1/y_t$, we get

$$2 - \alpha = (d + \mu y_3)\nu$$

This differs from the Josephson hyperscaling law, which is obtained by setting $\mu = 0$:

(c) In the Landau theory for $d > 4$, $\mu = 1$ and the crossover exponent is

$$y_3 = -\frac{d-4}{2} y_t$$

$$= -\frac{d-4}{2\nu}$$

The hyperscaling relation becomes

$$2 - \alpha = \left(d - \frac{d-4}{2\nu}\right)\nu$$

$$= d\nu - \frac{d}{2} + 2$$

This is satisfied for $\alpha = 0$ and $\nu = \frac{1}{2}$ for any d. Thus Landau theory satisfies this hyperscaling relation for $d > 4$.