Goldenfeld, Chapter 7

Exercise 7-1

The Landau free energy with a ϕ^n interaction term is

$$L = \int d^dx \left(\tfrac{1}{2} \nabla\phi\cdot\nabla\phi + \tfrac{1}{2} r_0 \phi^2 + \tfrac{1}{n!} u_n \phi^n \right)$$

(a) A stationary configuration $\phi(\vec{r}) = \phi_s$ that is constant in space must satisfy

$$r_0 \phi + \frac{1}{(n-1)!} u_n \phi^{n-1} = 0$$

If $r_0 < 0$, there is a nonzero solution

$$\phi_s = \left(\frac{(n-1)!\, |r_0|}{u_n} \right)^{\frac{1}{n-2}}$$

The correlation length can be obtained by setting $\phi(\vec{x}) = \phi_s + \psi(\vec{x})$ and expanding L to second order in ψ.

$$L = \int d^dx \left(\tfrac{1}{2} r_0 \phi_s^2 + \tfrac{1}{n!} u_n \phi_s^n \right)$$

$$+ \int d^dx \left(\tfrac{1}{2} \nabla\psi\cdot\nabla\psi + \tfrac{1}{2} r_0 \psi^2 + \frac{1}{2(n-2)!} u_n \phi_s^{n-2} \psi^2 \right)$$

$$= V \left(\tfrac{1}{2} r_0 \phi_s^2 - \frac{r_0}{n} \phi_s^2 \right)$$

$$+ \int d^dx \left(\tfrac{1}{2} \nabla\psi\cdot\nabla\psi + \tfrac{1}{2} \xi^{-2} \psi^2 \right)$$

The correlation length is given by

$$\xi^{-2} = r_0 + \frac{1}{(n-2)!} u_n \phi_s^{n-2}$$

$$= r_0 + (n-1)|r_0|$$

$$= (n-2)|r_0|$$

The Ginsburg criterion is

$$\int d^d r\, G(r) \ll \int d^d r \langle \phi^2 \rangle$$

The left side can be evaluated using the steps in Eq. (6.63), with $\gamma = 1$ and $2at = \xi^{-2}$. The result is

$$\int d^d r\, G(r) = kT \xi^2$$

We can approximate the right side by setting $\langle \phi^2 \rangle = \phi_s^2$ and assuming that the integral has support only in a volume proportional to ξ^d:

$$\int d^d r \langle \phi^2 \rangle \approx \xi^d \cdot \phi_s^2$$

Thus the Ginsburg criterion reduces to

$$kT_c\, \xi^2 \ll \xi^d \cdot \phi_s^2$$

Inserting our expressions for ξ and ϕ_s, this becomes

$$kT_c \cdot \frac{1}{(n-2)|r_0|} << \left(\frac{1}{(n-2)|r_0|}\right)^{d/2} \left(\frac{(n-1)!\,|r_0|}{u_n}\right)^{\frac{2}{n-2}}$$

Ignoring numerical factors, the reduces to

$$|r_0|^{\frac{d-2}{2} - \frac{2}{n-2}} << \frac{1}{kT_c\,(u_n)^{2/(n-2)}}$$

We assume $r_0 = at$. The Ginsburg criterion is satisfied as $t \to 0$ only if d is larger than the critical dimension

$$d = 2 + \frac{4}{n-2}$$

$$= \frac{2n}{n-2}$$

(b) The tricritical exponents in exercise 5.2 were derived from a Landau free energy corresponding to the Landau-Ginsburg Lagrangian in part (a) with $n = 6$. The critical dimension is $d = 3$. Thus the exponents will be correct only for $d \geq 3$.

(c) I did not finish this part