Goldenfeld, Chapter 5

Exercise 5-1

(a) The general expression for the Gibbs free energy is

$$\Gamma[m] = -\frac{1}{2}\sum_{ij} m_i J_{ij} m_j - \frac{1}{\beta}\sum_i \left(\log\frac{2}{\sqrt{1-m_i^2}} - \frac{1}{2} m_i \log\frac{1+m_i}{1-m_i} \right)$$

For the Ising model with uniform magnetization, the first term reduces to

$$-\frac{1}{2}\sum_{ij} m_i J_{ij} m_j = -\frac{1}{2}\sum_i m(zJm) = -\frac{1}{2}NzJm^2$$

Thus the Gibbs free energy reduces to

$$\Gamma(m) = -\frac{1}{2}NzJm^2 - \frac{N}{\beta}\left(\log\frac{2}{\sqrt{1-m^2}} - \frac{1}{2}m\log\frac{1+m}{1-m} \right)$$

The expansion to 4th order in m is

$$\Gamma(m) \approx -\frac{1}{2}NzJm^2 - \frac{N}{\beta}\left(\log 2 - \frac{1}{2}m^2 - \frac{1}{12}m^4 \right)$$

The magnetic field is

$$H = \frac{\partial\Gamma}{\partial m}$$

$$\simeq -NzJm - \frac{N}{\beta}\left(-m - \frac{1}{3}m^3 \right)$$

$$= N\left[\left(-zJ + \frac{1}{\beta} \right) m + \frac{1}{3\beta}m^3 \right]$$

For zero field ($H=0$), there are either 1 or 3 solutions.

If $zJ < 1/\beta$, the only solution is $m=0$.

If $zJ > \frac{1}{\beta}$, there are three solutions
$m=0$, with Gibbs free energy $\Gamma = -\frac{N}{\beta}\log 2$
and opposite sign solutions satisfying
$m^2 = 3\beta(zJ - 1/\beta)$, with Gibbs free energy.

$$\Gamma = -\tfrac{1}{2}NzJ\cdot 3\beta(zJ-1/\beta)$$

$$-\frac{N}{\beta}\left[\log 2 - \tfrac{1}{2}3\beta(zJ-\tfrac{1}{\beta}) - \tfrac{1}{12}\left(3\beta(zJ-1/\beta)\right)^2\right]$$

$$= -\frac{N}{\beta}\log 2 - \tfrac{1}{2}N\cdot 3\beta(zJ-1/\beta)^2 + \frac{N}{12\beta}9\beta^2(zJ-1/\beta)^2$$

$$= -\frac{N}{\beta}\log 2 - \tfrac{3}{4}N\beta(zJ-\tfrac{1}{\beta})^2$$

The opposite sign solution have lower Gibbs free energy, so they are the correct solutions.

Thus there is a transition at $kT = zJ$ from a state with $m=0$ to a state with nonzero magnetization.

The magnetization at 0 field is

$$m = \sqrt{\frac{3}{kT}(zJ-kT)}$$

$$= \sqrt{\frac{3}{kT}}\,(T_c - T)^{1/2}$$

Thus the critical exponent β defined by $m \propto (T_c - T)^\beta$ as $T \to T_c^-$ with $H=0$ is $\beta = \frac{1}{2}$

At the critical temperature, the magnetization satisfies

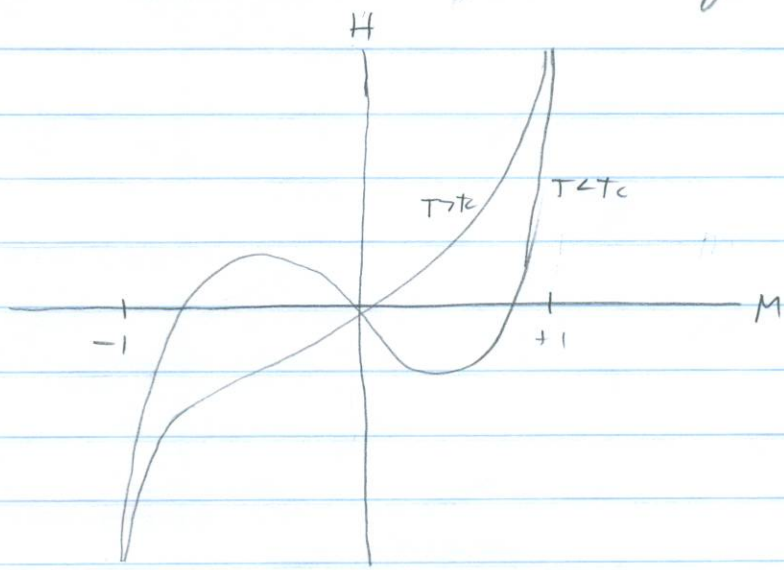
$$H = \frac{N}{3\beta} m^3$$

Thus the critical exponent δ defined by $m \propto H^{1/\delta}$ as $H \to 0$ with $T = T_c$ is $\delta = 3$.

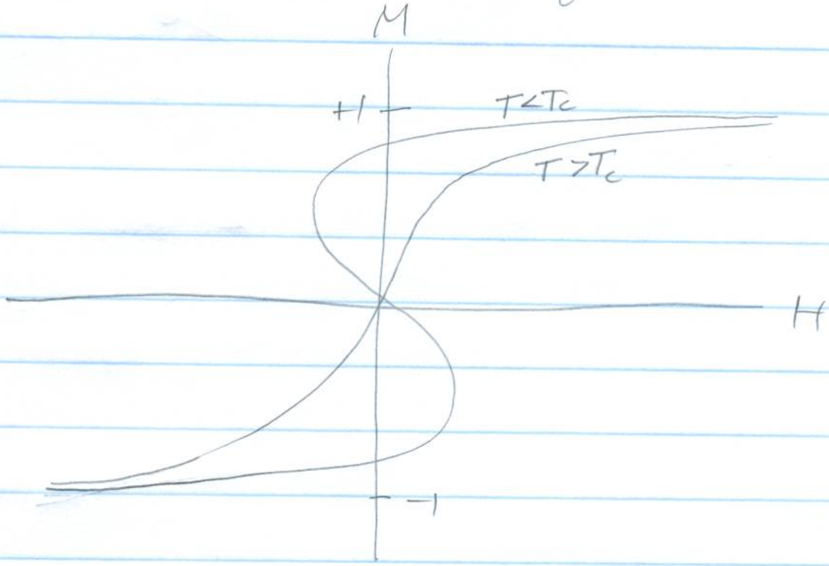
b. The function $H(m)$ in the mean field approximation is

$$H = \frac{1}{2\beta} \log \frac{1+m}{1-m} - zJm$$

Here is a sketch of the function for $T < T_c$ and $T > T_c$



The corresponding curves for $M(H)$ are



For $T<T_c$, the curves in the quadrant $M>0, H<0$ and in the quadrant $M<0, H>0$ are unphysical.

We define the function $L'(m,\hat{H})$ by

$$L'(m,\hat{H}) = \Gamma(m) - Nm\hat{H}$$

The minimum with respect to m satisfies

$$\frac{\partial}{\partial m} L'(m,\hat{H}) = 0$$

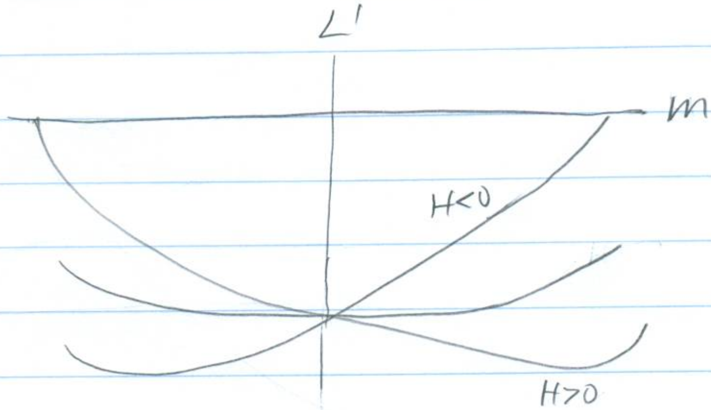
$$\text{or} \quad \frac{\partial \Gamma}{\partial m} - N\hat{H} = 0$$

The equation of state is

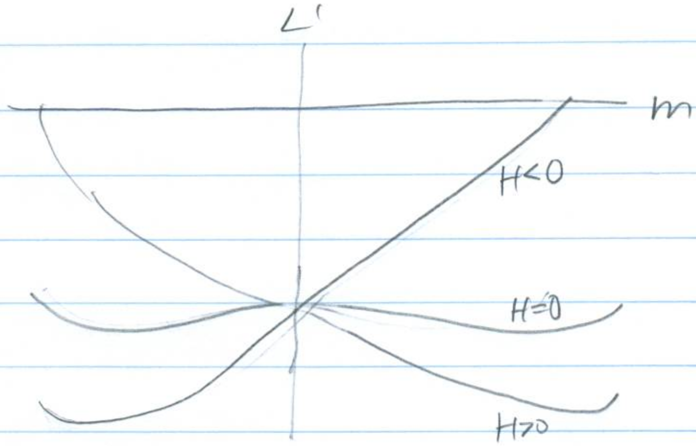
$$\frac{\partial \Gamma}{\partial m} - N H(m) = 0$$

Thus the variational equation implies the equation of state $H(m) = \hat{H}$.

The form of $L'(m, H)$ as a function of m above the transition is



Its form below the transition is



For $H>0$, the minimum always satisfies $m>0$
For $H<0$, the minimum always satisfies $m<0$.
Thus the unphysical portions of the curves for $m(H)$ are eliminated

Goldenfeld, Chapter 5

Exercise 5-2

The Landau free energy density is

$$\mathcal{L} = \frac{1}{2}a\eta^2 + \frac{1}{4}b\eta^4 + \frac{1}{6}c\eta^6 - h\eta$$

where $a = a_1 t + a_2 p$, $b = b_1 t + b_2 p$

a) The variational equation:

$$a\eta + b\eta^3 + c\eta^5 - h = 0$$

If $h=0$ and $a<0$, the solution η_s must be nonzero. It therefore satisfies

$$a + b\eta^2 + c\eta^4 = 0$$

The positive solution to the quadratic equation for η^2 is

$$\eta_s^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

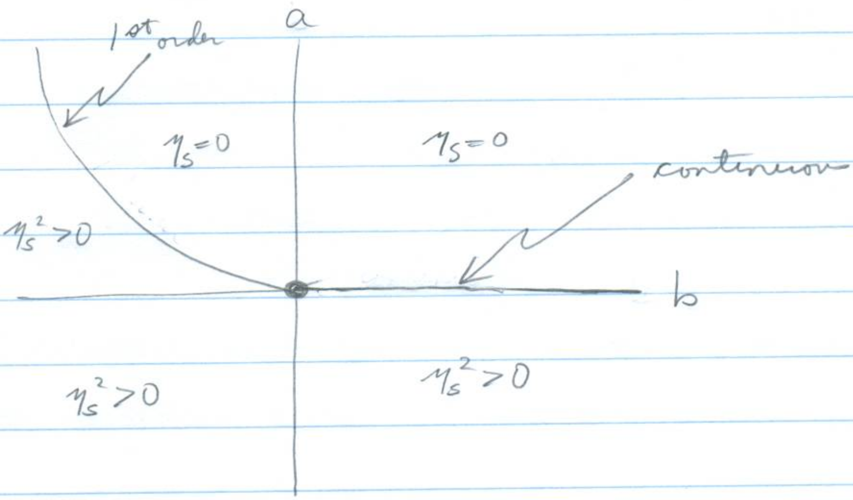
b) If $h=0$, $a>0$, and $b>0$, $\mathcal{L}$ is minimized by $\eta = 0$.

c) If $h=0$, $a>0$, and $b<0$, the solution is nonzero provided $b^2>4ac$:

$$\eta_s^2 = \frac{-b+\sqrt{b^2-4ac}}{2c}$$

If $b^2<4ac$, the only real solution is $\eta_s=0$

d) The phase diagram in the a-b plane is

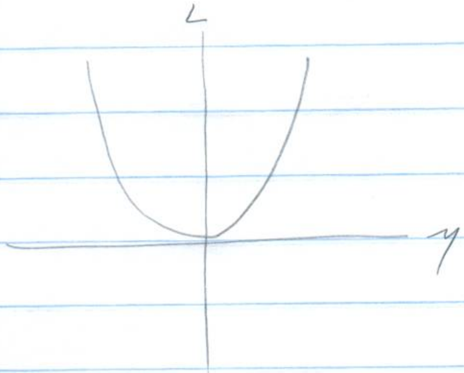


Above the b-axis with $b>0$, $\eta_s=0$. As you approach the line from below $\eta_s \to 0$. Thus the line is a continuous transition.

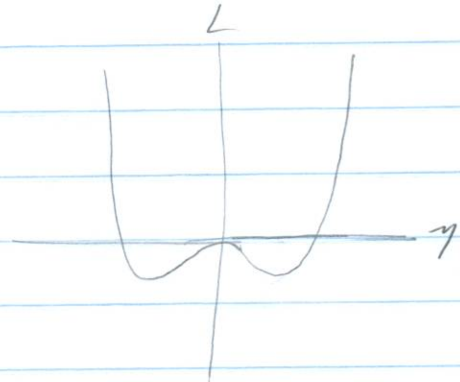
Across the line $b^2=4ac$ with $b<0$, η_s^2 changes discontinuously from $\eta_s^2=-b/2c$ to $\eta_s=0$.
So this line is a first-order transition.

Here are sketches of the Landau free energy in the various regions:

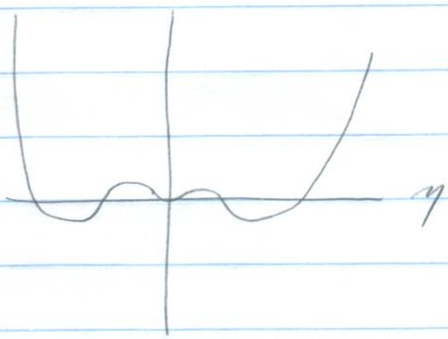
$a>0, b>0$



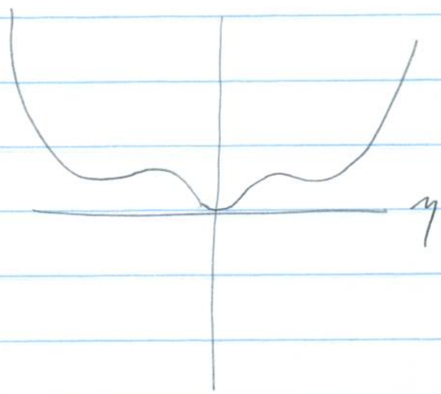
$a<0$, $b>0$ or $b<0$



$b<0$, $a<b^2/4c$



$b<0$, $a>b^2/4c$



The point $a=0, b=0$ may be called a tricritical point because it can be reached by tuning the 3 variables t, p, and h to 0 simultaneously

e) Along the line $b=0$, the variational equation is

$$a\eta + c\eta^5 - h = 0$$

At $h=0$ with $a<0$, the solution is

$$\eta = \left(-\frac{a}{c}\right)^{1/4}$$

$$= \left(-\frac{a_1 t + a_2 p}{c}\right)^{\frac{1}{4}}$$

This implies that $\beta = \frac{1}{4}$

At $p=t=0$, the solution is

$$\eta = \left(\frac{h}{c}\right)^{1/5}$$

Therefore $\delta = 5$

The susceptibility at $h=0$ is obtained by differentiating the variational equation with respect to h and then setting $h=0$:

$$\left(a + 5c\eta^4\right)\frac{\partial \eta}{\partial h} - 1 = 0$$

$$\frac{\partial \eta}{\partial h} = \frac{1}{a+5c\eta^4} = \frac{1}{a} \qquad a>0$$

$$= \frac{1}{-4a} \qquad a>0$$

Since $a = a_1 t + a_2 p_1$, the critical exponents are $\gamma = \gamma' = 1$.

The Landau free energy at 0 field is

$$\mathcal{L} = \tfrac{1}{2}a\eta^2 + \tfrac{1}{6}c\eta^6$$

$$= \tfrac{1}{2}\eta^2\left(a + \tfrac{1}{3}c\eta^4\right)$$

For $a>0$, $\eta=0$ so $\mathcal{L}=0$
For $a<0$, $\eta = (-a/c)^{1/4}$ so

$$\mathcal{L} = \tfrac{1}{2}\left(-\tfrac{a}{c}\right)^{1/2}\left[a + \tfrac{1}{3}c\left(-\tfrac{a}{c}\right)\right]$$

$$= -\tfrac{1}{3}c\left(-\tfrac{a}{c}\right)^{3/2}$$

$$= -\tfrac{1}{3}c\left(-\frac{a_1 t + a_2 p}{c}\right)^{3/2}$$

The heat capacity is the second derivative with respect to t, so

$$C \sim \left(-\frac{a_1 t + a_2 p}{c}\right)^{-\frac{1}{2}}$$

Since $C=0$ for $a<0$, the critical exponent are

$$\alpha = 0, \quad \alpha' = \tfrac{1}{2}$$

I did not have any expectations for ν.
Chapter 9 derives the Josephson scaling law:

$$\nu = \frac{2-\alpha}{d}$$

It predicts $\nu = 2/d$.

(f) If $b > 0$, the η^6 term can be ignored compared to the η^4 term for sufficiently small η. The Landau free energy density reduces to

$$\mathcal{L} = \frac{1}{2}a\eta^2 + \frac{1}{4}b\eta^4 - h\eta$$

This predicts ordinary critical behavior

The crossover to ordinary critical behavior will occur when the η^4 and η^6 terms are comparable.

$$\frac{1}{4}b\eta^4 \sim \frac{1}{6}c\eta^6$$

The scale of η can be obtained by demanding that the η^2 and η^4 terms are comparable.

$$\frac{1}{2}|a|\eta^2 \sim \frac{1}{4}b\eta^4$$

Combining these two relations, we get

$$\eta^2 \sim \frac{3}{2}\frac{b}{c} \sim \frac{2|a|}{b}$$

This implies $b^2 \sim |a|c$