Goldenfeld, Chapter 3

Exercise 3-1

(a) The transfer matrix T is

$$T = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}$$

We express the diagonalizing matrix as

$$S = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

The matrix that should be diagonal is

$$S^{-1}TS = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} ce^{h+K} + se^{-K} & -se^{h+K} + ce^{-K} \\ ce^{-K} + se^{-h+K} & -se^{-K} + ce^{-h+K} \end{pmatrix}$$

$$= \begin{pmatrix} c^2e^{h+K} + 2sce^{-K} + s^2e^{-h+K} & (c^2-s^2)e^{-K} - sc\left(e^{h+K} - e^{-h+K}\right) \\ (c^2-s^2)e^{-K} - sc\left(e^{h+K} - e^{-h+K}\right) & c^2e^{-h+K} - 2sce^{-K} + s^2e^{h+K} \end{pmatrix}$$

The off-diagonal elements are 0 if

$$(c^2 - s^2) e^{-K} = sc\, e^{K} (e^{h} - e^{-h})$$

$$\cos(2\phi)\, e^{-K} = \tfrac{1}{2} \sin(2\phi)\, e^{K}\, 2 \sinh h$$

$$\frac{1}{\tan(2\phi)} = e^{2K} \sinh h$$

(b) The expectation value of the spin at site i is

$$\langle S_i \rangle = \frac{1}{Z_N} \operatorname{tr}\left((T)^{i-1} \sigma_z (T)^{N-i+1} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(\sigma_z (T)^{N} \right)$$

Inserting $SS^{-1} = 1$ between each matrix, this becomes

$$\langle S_i \rangle = \frac{1}{Z_N} \operatorname{tr}\left(\sigma_z S (S^{-1} T S)^{N} S^{-1} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(S^{-1} \sigma_z S (T')^{N} \right)$$

The first matrix inside the trace is

$$S^{-1} \sigma_z S = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} c & -s \\ -s & -c \end{pmatrix}$$

$$= \begin{pmatrix} c^2 - s^2 & -2sc \\ -2sc & -c^2 + s^2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ -\sin(2\phi) & -\cos(2\phi) \end{pmatrix}$$

The expectation value of the spin reduces to

$$\langle S_i \rangle = \frac{1}{Z_N} \operatorname{tr}\left(\begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ -\sin(2\phi) & -\cos(2\phi) \end{pmatrix} \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \right)$$

$$= \frac{\cos(2\phi)\,\lambda_1^N - \cos(2\phi)\,\lambda_2^N}{\lambda_1^N + \lambda_2^N}$$

$$= \cos(2\phi)\,\frac{\lambda_1^N - \lambda_2^N}{\lambda_1^N - \lambda_2^N}$$

In the thermodynamic limit, $\left(\frac{\lambda_2}{\lambda_1}\right)^N \to 0$ and the reduces to

$$\langle S_i \rangle \longrightarrow \cos(2\phi)$$

We can calculate $\langle S_i S_j \rangle$ in a similar way:

$$\langle S_i S_j \rangle = \frac{1}{Z_N} \operatorname{tr}\left((T)^{i-1} \sigma_z (T)^{j-i} \sigma_z (T)^{N-j+1} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(\sigma_z (T)^{j-i} \sigma_z (T)^{N-j+i} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(\sigma_z S (S^{-1} T S)^{j-i} S^{-1} \sigma_z S (S^{-1} T S)^{N-j+i} S^{-1} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(S^{-1} \sigma_z S (T')^{j-i} S^{-1} \sigma_z S (T')^{N-j+i} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(\begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ -\sin(2\phi) & -\cos(2\phi) \end{pmatrix} \begin{pmatrix} \lambda_1^{j-i} & 0 \\ 0 & \lambda_2^{j-i} \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ -\sin(2\phi) & -\cos(2\phi) \end{pmatrix} \begin{pmatrix} \lambda_1^{N-j+i} & 0 \\ 0 & \lambda_2^{N-j+i} \end{pmatrix} \right)$$

$$= \frac{1}{Z_N} \operatorname{tr}\left(\begin{pmatrix} \cos(2\phi)\lambda_1^{j-i} & -\sin(2\phi)\lambda_2^{j-i} \\ -\sin(2\phi)\lambda_1^{j-i} & -\cos(2\phi)\lambda_2^{j-i} \end{pmatrix} \begin{pmatrix} \cos(2\phi)\lambda_1^{N-j+i} & -\sin(2\phi)\lambda_2^{N-j+i} \\ -\sin(2\phi)\lambda_1^{N-j+i} & -\cos(2\phi)\lambda_2^{N-j+i} \end{pmatrix} \right)$$

$$= \frac{1}{\lambda_1^N + \lambda_2^N} \left[\cos^2(2\phi)\lambda_1^N + \sin^2(2\phi)\lambda_1^N \left(\frac{\lambda_2}{\lambda_1}\right)^{j-i} \right.$$

$$\left. + \sin^2(2\phi)\lambda_2^N \left(\frac{\lambda_1}{\lambda_2}\right)^{j-i} + \cos^2(2\phi)\lambda_2^N \right]$$

In the thermodynamic limit, this reduces to

$$\langle S_i S_j \rangle \longrightarrow \cos^2(2\phi) + \sin^2(2\phi)\left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}$$

The 2-point correlation function is therefore

$$G(i,j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

$$= \sin^2(2\phi)\left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}$$

(c) The isothermal susceptibility is

$$\chi_T = \left(\frac{\partial M}{\partial H}\right)_T$$

The expression for M in Eq. (3.59) is

$$M = \frac{\sinh h}{\sqrt{\sinh^2 h + w^2}}$$

where $h = \beta H$ and $w = e^{-2K}$. The derivative is

$$\chi_T = \beta \frac{\partial}{\partial h} M$$

$$= \beta\left(\frac{\cosh h}{\sqrt{\sinh^2 h + w^2}} - \frac{1}{2}\frac{\sinh h}{(\sinh^2 h + w^2)^{3/2}} \cdot 2 \sinh h \cosh h\right)$$

$$= \beta \frac{\cosh h(\cancel{\sinh^2 h} + w^2) - \cancel{\sinh^2 h \cosh h}}{(\sinh^2 h + w^2)^{3/2}}$$

$$= \beta \frac{w^2 \cosh h}{(\sinh^2 h + w^2)^{3/2}}$$

The expression for the isothermal susceptibility in Eq. (3.160) is

$$\chi_T = \frac{1}{kT}\sum_j G(i, i+j)$$

$$= \frac{1}{kT}\sum_{j=-\infty}^{+\infty} \sin^2(2\phi)\left(\frac{\lambda_1}{\lambda_2}\right)^{|j|}$$

$$= \frac{1}{kT}\sin^2(2\phi)\left[1 + 2\sum_{j=1}^{\infty}\left(\frac{\lambda_1}{\lambda_2}\right)^j\right]$$

$$= \frac{1}{kT} \sin^2(2\phi) \left[1 + 2 \frac{\lambda_1/\lambda_2}{1-\lambda_1/\lambda_2} \right]$$

$$= \frac{1}{kT} \sin^2(2\phi) \frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}$$

$$= \frac{1}{kT} \sin^2(2\phi) \frac{e^K \cdot 2\cosh h}{e^K 2\sqrt{\sinh^2 h + e^{-4K}}}$$

$$= \frac{1}{kT} \sin^2(2\phi) \frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}$$

The trigonometric prefactor is

$$\sin^2(2\phi) = \frac{\sin^2(2\phi)}{\sin^2(2\phi)+\cos^2(2\phi)}$$

$$= \frac{1}{1 + 1/\tan^2(2\phi)}$$

$$= \frac{1}{1 + (e^{2K}\sinh h)^2}$$

$$= \frac{1}{e^{4K}(\sinh^2 h + e^{-4K})}$$

Thus our second expression for the susceptibility is

$$\chi_T = \frac{1}{kT} \frac{e^{-4K}\cosh h}{(\sinh^2 h + e^{-4K})^{3/2}}$$

This agrees with the result from differentiating M.

(d) The partition function with free boundary conditions is

$$Z_N = \sum_{s_1} \cdots \sum_{s_N} e^{h(s_1+\cdots+s_N)+K(s_1s_2+\cdots+s_{N-1}s_N)}$$

$$= \sum_{s_1} \cdots \sum_{s_N} e^{\frac{h}{2}(s_1+s_2)+Ks_1s_2} e^{\frac{h}{2}(s_2+s_3)+Ks_2s_3} \times \cdots$$

$$\times e^{\frac{h}{2}(s_{N-1}+s_N)+Ks_{N-1}s_N} e^{\frac{h}{2}(s_N+s_1)}$$

$$= \mathrm{tr}\left((T)^{N-1} V\right)$$

where $T = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}$ and $V = \begin{pmatrix} e^{h} & 1 \\ 1 & e^{-h} \end{pmatrix}$

Inserting $SS^{-1} = 1$ between every pair of matrices, this can be written

$$Z_N = \mathrm{Tr}\left((S^{-1}TS)^{N-1} S^{-1}VS\right)$$

$$= \mathrm{Tr}\left((T')^{N-1} V'\right)$$

where $T' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$V' = S^{-1}VS$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} e^{h} & 1 \\ 1 & e^{-h} \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} ce^{h}+s & -se^{h}+c \\ c+e^{-h}s & -s+ce^{-h} \end{pmatrix}$$

$$= \begin{pmatrix} c(ce^h+s)+s(c+e^{-h}s) & c(-se^h+c)+s(-s+ce^{-h}) \\ -s(ce^h+s)+c(c+e^{-h}s) & -s(-se^h+c)+c(-s+ce^{-h}) \end{pmatrix}$$

$$= \begin{pmatrix} c^2e^h+2sc+s^2e^{-h} & -cse^h+c^2-s^2+cse^{-h} \\ -sce^h+c^2-s^2+cse^{-h} & -s^2e^h-2cs+c^2e^{-h} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{ch}\,h+\cos(2\phi)\,\mathrm{sh}\,h+\sin(2\phi) & \cos(2\phi)-\sin(2\phi)\,\mathrm{sh}\,h \\ \cos(2\phi)-\sin(2\phi)\,\mathrm{sh}\,h & \cos(2\phi)\,\mathrm{ch}\,h-\mathrm{sh}\,h-\sin(2\phi) \end{pmatrix}$$

The partition function is

$$Z_N = \lambda_1^{N-1}\left[\mathrm{ch}\,h+\cos(2\phi)\,\mathrm{sh}\,h+\sin(2\phi)\right]$$
$$+\lambda_2^{N-1}\left[\cos(2\phi)\,\mathrm{ch}\,h-\mathrm{sh}\,h-\sin(2\phi)\right]$$

The free energy is

$$F_N = -kT\log Z_N$$

$$= -kT\Big[(N-1)\log\lambda_1+\log\left[\mathrm{ch}\,h+\cos(2\phi)\,\mathrm{sh}\,h+\sin(2\phi)\right]$$
$$+\log\Big(1+\left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\frac{\cos(2\phi)\,\mathrm{ch}\,h-\mathrm{sh}\,h-\sin(2\phi)}{\mathrm{ch}\,h+\cos(2\phi)\,\mathrm{sh}\,h+\sin(2\phi)}\Big)\Big]$$

$$= Nf_b+f_s+F_{fs}$$

where $f_b = -kT\log\lambda_1$

$$f_s = -kT\log\frac{\mathrm{ch}\,h+\cos(2\phi)\,\mathrm{sh}\,h+\sin(2\phi)}{\lambda_1}$$

The leading contribution to the remaining as $N \to \infty$ is

$$F_{fs} \approx -kT \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1} \frac{\cos(2\phi)\,\mathrm{ch}\,h - \mathrm{sh}\,h - \sin(2\phi)}{\mathrm{ch}\,h + \cos(2\phi)\,\mathrm{sh}\,h + \sin(2\phi)}$$

This decreases exponentially as $e^{-N \log(\lambda_2/\lambda_1)}$

(e) The trigonometric functions in f_s can be expressed as

$$\sin(2\phi) = \frac{w}{\sqrt{\mathrm{sh}^2 h + w^2}}$$

$$\cos(2\phi) = \frac{\mathrm{sh}\,h}{\sqrt{\mathrm{sh}^2 h + w^2}}$$

Our expression for the surface free energy therefore reduce to

$$f_s = -kT \log \frac{\mathrm{ch}\,h + \frac{\mathrm{sh}\,h}{\sqrt{\mathrm{sh}^2 h + w^2}}\,\mathrm{sh}\,h + \frac{w}{\sqrt{\mathrm{sh}^2 h + w^2}}}{e^K\left(\mathrm{ch}\,h + \sqrt{\mathrm{sh}^2 h + w^2}\right)}$$

$$= -kT \log \frac{\mathrm{ch}\,h\sqrt{\mathrm{sh}^2 h + w^2} + \mathrm{sh}^2 h + w}{e^K\left(\mathrm{ch}\,h + \sqrt{\mathrm{sh}^2 h + w^2}\right)\sqrt{\mathrm{sh}^2 h + w^2}}$$

where $w = e^{-2K}$. In the case $h = 0$, this reduces to

$$f_s = -kT \log \frac{2w}{e^K(1+w)w}$$

$$= -kT \log \frac{1}{\cosh K}$$

The surface free energy can also be obtained from the difference between the free energy with free boundary conditions and periodic boundary conditions

$$f_s = \lim_{N\to\infty}\left(F_N^{\text{free}} - F_N^{\text{periodic}}\right)$$

$$= -kT \lim_{N\to\infty}\Big((N-1)\log(2\cosh K) + \log 2$$

$$- \log\left[(2\cosh K)^N + (2\sinh K)^N\right]\Big)$$

$$= -kT \lim_{N\to\infty}\Big((N-1)\log(2\cosh K) + \log 2$$

$$- \left[N\log(2\cosh K) + \log\left(1 + (\tanh K)^N\right)\right]\Big)$$

$$= -kT\left(-\log(2\cosh K) + \log 2\right)$$

$$= -kT \log\frac{1}{\cosh K}$$

This agrees with the previous expression.

Goldenfeld, Chapter 3

Exercise 3-2

(a) The spin variable can be specified by a single index: $S_n = S_{1n} = S_{2n}$. The Hamiltonian is given by

$$-\beta H_\Omega = K\sum_{n=1}^{N}\left(S_{1n}S_{2n} + S_{1n}S_{1,n+1}\right)$$

$$= K\sum_{n=1}^{N}\left(S_n^2 + S_n S_{n+1}\right)$$

$$= K\sum_{n=1}^{N}\left(1 + S_n S_{n+1}\right)$$

The exponential of the Hamiltonian is

$$e^{-\beta H_\Omega} = \prod_{n=1}^{N} e^{K(1+S_nS_{n+1})}$$

$$= \prod_{n=1}^{N} (T)_{S_n, S_{n+1}}$$

where the transfer matrix T is the 2×2 matrix:

$$T = \begin{pmatrix} e^{2K} & 1 \\ 1 & e^{2K} \end{pmatrix}$$

The eigenvalues λ_1 and λ_2 of the matrix satisfy

$$(e^{2K} - \lambda)^2 - 1 = 0$$

The solutions of this quadratic equation are

$$\lambda_{1,2} = e^{2K} \pm 1$$

This can be written

$$\lambda_1 = x^2 + 1 \quad \lambda_2 = x^2 - 1$$

where $x = e^K$

(b) The Hamiltonian for $M=2$ is given by

$$-\beta H_\Omega = K \sum_{n=1}^{N} \left(S_{1n} S_{2n} + S_{1n} S_{1n+1} + S_{2n} S_{3n} + S_{2n} S_{2n+1} \right)$$

$$= K \sum_{n=1}^{N} \left(S_{1n} S_{1,n+1} + S_{2n} S_{2,n+1} + 2 S_{1n} S_{2n} \right)$$

$$= K \sum_{n=1}^{N} \left(S_{1n} S_{1,n+1} + S_{2n} S_{2,n+1} + S_{1n} S_{2n} + S_{1,n+1} S_{2,n+1} \right)$$

$$= K \sum_{n=1}^{N} \left(S_{1,n} + S_{2,n+1} \right) \left(S_{1,n+1} + S_{2n} \right)$$

Its exponential is

$$e^{-\beta H_\Omega} = \prod_{n=1}^{N} e^{K(S_{1,n} + S_{2,n+1})(S_{1,n+1} + S_{2n})}$$

$$= \prod_{n=1}^{N} (T)_{(S_{1n}, S_{2n}),(S_{1,n+1}, S_{2,n+1})}$$

where the transfer matrix T is a 4×4 matrix.

(c) If we choose the ordering of the state (S_{1n}, S_{2n}) as $(+1,+1)$, $(+1,-1)$, $(-1,+1)$, $(-1,-1)$, the transfer matrix is

$$T = \begin{pmatrix} e^{4K} & 1 & 1 & 1 \\ 1 & 1 & e^{-4K} & 1 \\ 1 & e^{-4K} & 1 & 1 \\ 1 & 1 & 1 & e^{-4K} \end{pmatrix}$$

$$= \begin{pmatrix} x^4 & 1 & 1 & 1 \\ 1 & 1 & x^{-4} & 1 \\ 1 & x^{-4} & 1 & 1 \\ 1 & 1 & 1 & x^4 \end{pmatrix}$$

(d) The eigenvalues of this matrix can be found easily using Mathematica:

$$x^4-1,\ 1-x^{-4},\ \tfrac{1}{2}(x^2+x^{-2})^2 \pm \tfrac{1}{2}\sqrt{x^8+14+x^{-8}}$$

By plotting the eigenvalues as function of $x = e^K$, we can easily verify that the two largest ones are

$$\lambda_1 = \tfrac{1}{2}(x^2+x^{-2})^2 + \tfrac{1}{2}\sqrt{x^8+14+x^{-8}}$$

$$\lambda_2 = x^4-1$$

Goldenfeld, Chapter 3

Exercise 3-3

(a) The argument of the exponential can be written

$$-\tfrac{1}{2} x_i A_{ij} x_j + x_i B_i$$

$$= -\tfrac{1}{2}\left[x_i - (BA^{-1})_i\right] A_{ij} \left[x_j - (A^{-1}B)_j\right]$$

$$-\tfrac{1}{2}(BA^{-1})_i A_{ij} x_j - \tfrac{1}{2} x_i A_{ij} (A^{-1}B)_j$$

$$+\tfrac{1}{2}(BA^{-1})_i A_{ij} (A^{-1}B)_j + x_i B_i$$

$$= -\tfrac{1}{2}\left[x_i - (BA^{-1})_i\right] A_{ij} \left[x_j - (A^{-1}B)_j\right]$$

$$\cancel{-\tfrac{1}{2}(BA^{-1}A)_j x_j} \cancel{- \tfrac{1}{2} x_i (AA^{-1}B)_i}$$

$$+\tfrac{1}{2} B_i (A^{-1}AA^{-1})_{ij} B_j \cancel{+ x_i B_i}$$

$$= -\tfrac{1}{2}\left[x_i - (BA^{-1})_i\right] A_{ij} \left[x_j - (A^{-1}B)_j\right]$$

$$+\tfrac{1}{2} B_i (A^{-1})_{ij} B_j$$

Thus if we make the change of variables $y_i = x_i - (BA^{-1})_i$, the integral can be written

$$\prod_{i=1}^{N}\left(\int_{-\infty}^{\infty}\frac{dx_i}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}x_i A_{ij} x_j + x_i B_i\right)$$

$$= \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty}\frac{dy_i}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}y_i A_{ij} y_j + \tfrac{1}{2}B_i (A^{-1})_{ij} B_j\right)$$

$$= \exp\left(\tfrac{1}{2}B_i (A^{-1})_{ij} B_j\right)\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{dy_i}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}y_i A_{ij} y_j\right)$$

There is an orthogonal matrix $\mathcal{O}$ that diagonalizes A:

$$\mathcal{O} A \mathcal{O}^T = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_N \end{pmatrix}$$

The argument of the exponential can be written

$$-\tfrac{1}{2}y_i A_{ij} y_j = -\tfrac{1}{2}y_i (\mathcal{O}^T\mathcal{O})_{ik} A_{k\ell} (\mathcal{O}^T\mathcal{O})_{\ell j} y_j$$

$$= -\tfrac{1}{2}y_i (\mathcal{O}^T)_{ik} (\mathcal{O}A\mathcal{O}^T)_{k\ell}\, \mathcal{O}_{\ell j} y_j$$

$$= -\tfrac{1}{2}\sum_{k=1}^{N} a_k\, y_i (\mathcal{O}^T)_{ik}\, \mathcal{O}_{\ell j} y_j$$

Thus if we make the change of variables $z_i = \mathcal{O}_{ij} y_j$, the remaining integral can be written

$$\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{dy_i}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}y_i A_{ij} y_j\right)$$

$$= \left(\prod_{i=1}^{N}\int\frac{dz_i}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}\sum_{k} a_k z_k^2\right)$$

$$= \prod_{i=1}^{N}\int\frac{dz_i}{\sqrt{2\pi}}\, e^{-\frac{1}{2}a_i z_i^2}$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{a_i}}$$

$$= \frac{1}{\sqrt{\det A}}$$

Thus the original integral is

$$\prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{2\pi}} \right) \exp\left(-\tfrac{1}{2} x_i A_{ij} x_j + x_i B_i \right)$$

$$= \frac{1}{\sqrt{\det A}} \exp\left(+ \tfrac{1}{2} B_i (A^{-1})_{ij} B_j \right)$$

(b) To apply the identity to the Ising model, we choose $B_i = \beta S_i$ and $A^{-1} = \frac{1}{\beta} J$. The Hamiltonian is given by

$$-\beta H_{\Omega} = \tfrac{1}{2} \beta \sum_{i \neq j} J_{ij} S_i S_j + \beta \sum_i H_i S_i$$

$$= \tfrac{1}{2} \beta \sum_{ij} J_{ij} S_i S_j - \tfrac{1}{2} \beta \sum_i J_{ii} S_i^2 + \beta \sum_i H_i S_i$$

$$= \tfrac{1}{2} (\beta S_i) \left(\tfrac{1}{\beta} J_{ij} \right) (\beta S_j) + H_i (\beta S_i) - \tfrac{1}{2} \beta \operatorname{tr} J$$

Nonzero diagonal entries J_{ii} may be necessary in order to make the matrix J_{ij} invertible. The last term $-\frac{1}{2}\beta \operatorname{tr} J$ corresponds to spin-independent term $\frac{1}{2} \operatorname{tr} J$ in the energy.

(c) The partition function can be written

$$Z_\Omega = \sum_{S_1} \cdots \sum_{S_N} e^{-\beta H_\Omega}$$

$$= e^{-\frac{1}{2}\beta \operatorname{tr} J} \sum_{S_1} \cdots \sum_{S_N} \exp\left(\tfrac{1}{2}(\beta S_i)\left(\tfrac{1}{\beta} J_{ij}\right)(\beta S_j) + H_i(\beta S_i)\right)$$

$$= e^{-\frac{1}{2}\beta \operatorname{tr} J} \sum_{S_1} \cdots \sum_{S_N} \exp(\beta H_i S_i) \times \sqrt{\det(\beta J^{-1})}$$

$$\times \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \frac{d\psi_i}{\sqrt{2\pi}}\right) \exp\left(-\tfrac{1}{2}\psi_i (\beta J^{-1})_{ij} \psi_j + \psi_i (\beta S_i)\right)$$

$$= \sqrt{\det(\beta J^{-1})}\, e^{-\frac{1}{2}\beta \operatorname{tr} J}$$

$$\times \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \frac{d\psi_i}{\sqrt{2\pi}}\right) \exp\left(-\tfrac{1}{2}\beta \psi_i (J^{-1})_{ij} \psi_j\right)$$

$$\times \sum_{S_1} \cdots \sum_{S_N} e^{\beta(H_i S_i + \psi_i S_i)}$$

After shifting the integration variables by $\psi_i \longrightarrow \psi_i - H_i$ the sums reduce to

$$\sum_{S_1} \cdots \sum_{S_N} e^{\beta \psi_i S_i} = \prod_{i=1}^{N} \left(\sum_{S_i = \pm 1} e^{\beta \psi_i S_i}\right)$$

$$= \prod_{i=1}^{N} \left(e^{\beta \psi_i} + e^{-\beta \psi_i}\right)$$

$$= \prod_{i=1}^{N} 2\cosh(\beta \psi_i)$$

$$= \exp\left(\sum_i \log\left[2\cosh(\beta \psi_i)\right]\right)$$

Thus the partition function can be written

$$Z_\Omega = \sqrt{\det(\beta J^{-1})}\, e^{-\frac{1}{2}\beta \operatorname{tr} J}$$
$$\times \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \frac{d\psi_i}{\sqrt{2\pi}} \right) e^{-\beta S[\psi]}$$

where the action $S[\psi]$ is

$$S = \tfrac{1}{2}(\psi_i - H_i)(J^{-1})_{ij}(\psi_j - H_j) - \frac{1}{\beta}\sum_i \log[2\cosh(\beta\psi_i)]$$

(d) The variation of the action from varying ψ is

$$\delta S = \tfrac{1}{2}\delta\psi_i (J^{-1})_{ij}(\psi_j - H_j) + \tfrac{1}{2}(\psi_i - H_i)(J^{-1})_{ij}\delta\psi_j$$
$$- \frac{1}{\beta}\sum_i \frac{1}{2\cosh(\beta\psi_i)} 2\sinh(\beta\psi_i)\, \beta\psi_i$$
$$= \sum_i \delta\psi_i \left[(J^{-1})_{ij}(\psi_j - H_j) - \tanh(\beta\psi_i) \right]$$

The field $\bar{\Psi}_i$ that minimizes S therefore satisfies

$$(J^{-1})_{ij}(\bar{\Psi}_j - H_j) - \tanh(\beta\bar{\Psi}_i) = 0$$

The magnetization m_i at site i is

$$m_i = \langle S_i \rangle = -\frac{\partial}{\partial H_i} F$$

If we approximate the integral by by the value of the integrand at the minimum, we have

$$Z \approx e^{-\beta S[\bar{\Psi}]}$$

$$F \approx S[\bar{\Psi}]$$

The magnetization can therefore be approximated by

$$m_i \approx -\frac{\partial}{\partial H_i} S[\bar{\Psi}]$$

$$= + (J^{-1})_{ij} (\bar{\Psi}_j - H_j)$$

$$= \tanh(\beta \bar{\Psi}_i)$$

This can be inverted to express $\bar{\Psi}_i$ as a function of m_i:

$$\bar{\Psi}_i = \frac{1}{2\beta} \log \frac{1+m_i}{1-m_i}$$

The minimization equation for S can now be used to express H_i as a function of m_i:

$$\bar{\Psi}_j - H_j = \sum_i J_{ji} \tanh(\beta \bar{\Psi}_i)$$

$$\frac{1}{\beta} \log \frac{1+m_j}{1-m_j} - H_j = \sum_i J_{ji} m_i$$

$$H_j = \frac{1}{2\beta} \log \frac{1+m_j}{1-m_j} - \sum_k J_{jk} m_k$$

(e) The minimum of the action can now be expressed as a function of m

$$\bar{S} = \tfrac{1}{2}(\Psi_i - H_i)(J^{-1})_{ij}(\Psi_j - H_j) - \tfrac{1}{\beta}\sum_i \log\left[2\cosh(\beta\Psi_i)\right]$$

$$= \tfrac{1}{2}(J_{ik}m_k)(J^{-1})_{ij}(J_{jl}m_l)$$

$$- \tfrac{1}{\beta}\sum_i \log\left[2\cosh\left(\tfrac{1}{2}\log\tfrac{1+m_i}{1-m_i}\right)\right]$$

$$= \tfrac{1}{2}m_k J_{kl} m_l - \tfrac{1}{\beta}\sum_i \log\left(\sqrt{\tfrac{1+m_i}{1-m_i}} + \sqrt{\tfrac{1-m_i}{1+m_i}}\right)$$

$$= \tfrac{1}{2}m_i J_{ij} m_j - \tfrac{1}{\beta}\sum_i \log\frac{2}{\sqrt{1-m_i^2}}$$

The Legendre transform of the action is

$$\Gamma[m] = \bar{S} + H_i m_i$$

$$= \tfrac{1}{2}m_i J_{ij} m_j - \tfrac{1}{\beta}\sum_i \log\frac{2}{\sqrt{1-m_i^2}}$$

$$+ \sum_i \left(\tfrac{1}{2\beta}\log\tfrac{1+m_i}{1-m_i} - \sum_k J_{ik}m_k\right)m_i$$

$$= -\tfrac{1}{2}m_i J_{ij} m_j + \tfrac{1}{\beta}\sum_i\left[\tfrac{1}{2}m_i\log\tfrac{1+m_i}{1-m_i} - \log\frac{2}{\sqrt{1-m_i^2}}\right]$$

The variation of Γ with respect to m_i is

$$\delta\Gamma = -J_{ij}m_j + \tfrac{1}{\beta}\left[\tfrac{1}{2}\log\tfrac{1+m_i}{1-m_i} + \tfrac{1}{2}m_i\left(\tfrac{1}{1+m_i} + \tfrac{1}{1-m_i}\right)\right.$$

$$\left. + \tfrac{1}{2}\tfrac{1}{1-m_i^2}(-2m_i)\right]$$

$$= -J_{ij} m_j + \frac{1}{2\beta} \log \frac{1+m_i}{1-m_i}$$

Thus the equation of state is correctly given by

$$\frac{\partial \Gamma}{\partial m_i} = H_i$$