Schwartz 14.1*

We consider the following functional integral for a complex scalar field $\phi(x)$ with a complex source $J(x)$:

$$\int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\Big(i \int d^4x \int d^4y\, \phi^*(x)\, M(x,y)\, \phi(y)$$

$$+ i \int d^4x \big(J^*(x)\, \phi(x) + \phi^*(x)\, J(x) \big) \Big)$$

where the operator M satisfies $M(x,y)^* = M(y,x)$.

If we shift the field $\phi(x)$ by

$$\phi(x) \longrightarrow \phi(x) - \int d^4z\, M^{-1}(x,z)\, \phi(z)$$

the shift in its complex conjugate is

$$\phi^*(x) \longrightarrow \phi^*(x) - \int d^4z\, J^*(z)\, M^{-1}(z,x)$$

The functional integral is invariant under the shift. It is therefore equal to

$$\int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\Big(i \int d^4x \int d^4y \Big[\phi^*(x) - \int d^4w\, J^*(w)\, M^{-1}(w,x)\Big] M(x,y)$$

$$\times \Big[\phi(y) - \int d^4z\, M^{-1}(y,z)\, J(z)\Big]$$

$$+ i \int d^4x\, J^*(x) \Big[\phi(x) - \int d^4z\, M^{-1}(x,z)\, J(z)\Big]$$

$$+ i \int d^4x \Big[\phi^*(x) - \int d^4w\, J^*(w)\, M^{-1}(w,x)\Big] J(x) \Big)$$

$$= \int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\Big(i \int d^4x \int d^4y\, \phi^*(x)\, M(x,y)\, \phi(y) - i \int d^4w\, \cancel{J^*(w)}\, \phi(w)$$

$$- i \int d^4z\, \cancel{\phi^*(z)}\, J(z) + i \int d^4x \int d^4y\, J^*(x)\, M^{-1}(x,y)\, J(y)$$

$$+ i \int d^4x\, \cancel{J^*(x)}\, \phi(x) - i \int d^4x \int d^4z\, J^*(x)\, M^{-1}(x,z)\, J(z)$$

$$+ i \int d^4x\, \cancel{\phi^*(x)}\, J(x) - i \int d^4x \int d^4w\, J^*(w)\, M^{-1}(w,x)\, J(x) \Big)$$

$$= \int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\Big(i \int d^4x \int d^4y\, \phi^*(x)\, M(x,y)\, \phi(y)$$

$$- i \int d^4x \int d^4y\, J^*(x)\, M^{-1}(x,y)\, J(y) \Big)$$

$$= \exp\Big(-i \int d^4x \int d^4y\, J^*(x)\, M^{-1}(x,y)\, J(y) \Big)$$

$$\times \int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\Big(i \int d^4x \int d^4y\, \phi^*(x)\, M(x,y)\, \phi(y) \Big)$$

The remaining functional integral is a Gaussian integral. It is equal to $1/\det M$ multiplied by an infinite constant. Thus the complete functional integral is

$$N \frac{1}{\det M} \exp\Big(-i \int d^4x \int d^4y\, J^*(x)\, M^{-1}(x,y)\, J(y) \Big)$$

Schwartz 14.2abc

(a) The Lagrangian for scalar QED is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi$$

$$= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi$$

The action of the charge conjugation operator C on the scalar field is

$$C\phi(x) = \phi^*(x)$$

$$C\phi^*(x) = \phi(x)$$

For the term $-ieA_\mu(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)$ to be invariant, the action of C on the photon field must be

$$CA_\mu(x) = -A_\mu(x)$$

(b) The n-photon Green function is

$$G_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = \langle\Omega|T A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)|\Omega\rangle$$

$$= \frac{\int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}A\, e^{iS} A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)}{\int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}A\, e^{iS}}$$

If we make a charge conjugation transformation on the fields, the action is invariant and the measure is invariant, but the factors of the photon field in the numerator change

$$G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n)$$

$$= \frac{\int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}A \, e^{iS} \left(-A_{\mu_1}(x_1)\right) \cdots \left(-A_{\mu_n}(x_n)\right)}{\int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}A}$$

$$= (-1)^n \frac{\int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}A \, e^{iS} A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)}{\int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}A}$$

$$= (-1)^n G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n)$$

This implies Furry's Theorem

$$\boxed{G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n) = 0 \quad \text{if } n \text{ is odd}}$$

(c) It holds when the photons are off-shell. Since the Green function for on-shell photons is obtained by Fourier transforming in all n coordinates, amputating the propagators, and then putting the n momenta on-shell, it also holds if the photons are on-shell.

Schwartz 14.3

(a) The Fourier expansions of the field operator $\hat{\phi}(\vec{r},t)$ and the conjugate momentum operator $\hat{\pi}(\vec{r},t) = \frac{\partial}{\partial t}\hat{\phi}(\vec{r},t)$ at time $t=0$ are

$$\hat{\phi}(\vec{r},t=0) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(\hat{a}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{r}} + \hat{a}^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{r}}\right)$$

$$\hat{\pi}(\vec{r},t=0) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left((-i\omega_p)\hat{a}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{r}} + (i\omega_p)\hat{a}^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{r}}\right)$$

The Fourier transforms of these equations are

$$\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\,\hat{\phi}(\vec{r}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\left(\hat{a}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{r}} + \hat{a}^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{r}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[\hat{a}_{\vec{p}}\,(2\pi)^3\delta^3(\vec{p}-\vec{q}) + \hat{a}^\dagger_{\vec{p}}\,(2\pi)^3\delta^3(\vec{p}+\vec{q})\right]$$

$$= \frac{1}{\sqrt{2\omega_q}}\left(\hat{a}_{\vec{q}} + \hat{a}^\dagger_{-\vec{q}}\right)$$

$$\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\,\hat{\pi}(\vec{r}) = \int \frac{d^3p}{(2\pi)^3}\frac{-i\omega_p}{\sqrt{2\omega_p}}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\left(-\hat{a}_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{r}} + \hat{a}^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{r}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{-i\omega_p}{\sqrt{2\omega_p}}\left[-\hat{a}_{\vec{p}}\,(2\pi)^3\delta^3(\vec{p}-\vec{q}) + \hat{a}^\dagger_{\vec{p}}\,(2\pi)^3\delta^3(\vec{p}+\vec{q})\right]$$

$$= -\frac{i\omega_q}{\sqrt{2\omega_q}}\left(\hat{a}_{\vec{q}} - \hat{a}^\dagger_{-\vec{q}}\right)$$

By taking an appropriate linear combination, we can get an expression with only the annihilation operator $\hat{a}_{\vec{q}}$ on the right side.

$$\omega_q \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\,\hat{\phi}(\vec{r}) + i\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\,\hat{\pi}(\vec{r}) = \frac{2\omega_q}{\sqrt{2\omega_q}}\,\hat{a}_{\vec{q}}$$

The resulting expression for $\hat{a}_{\vec{p}}$ is

$$\hat{a}_p = \frac{1}{\sqrt{2\omega_p}}\int d^3r\, e^{-i\vec{p}\cdot\vec{r}}\left(\omega_p\,\hat{\phi}(\vec{r}) + i\,\hat{\pi}(\vec{r})\right)$$

(b) The operators $\hat{\phi}(\vec{r})$ and $\hat{\pi}(\vec{r})$ satisfy the canonical commutation relation

$$\left[\hat{\phi}(\vec{r}),\ \hat{\pi}(\vec{r}\,')\right] = i\,\delta^3(\vec{r}-\vec{r}\,')$$

Suppose $|\phi\rangle$ is a simultaneous eigenstate of $\hat{\phi}(\vec{r})$ for all $\vec{r}$:

$$\hat{\phi}(\vec{r})\,|\phi\rangle = \phi(\vec{r})\,|\phi\rangle$$

The commutation relation applied to $|\Phi\rangle$ is

$$\hat{\phi}(\vec{r})\,\hat{\pi}(\vec{r}\,')\,|\phi\rangle - \hat{\pi}(\vec{r}\,')\,\hat{\phi}(\vec{r})\,|\phi\rangle = i\,\delta^3(\vec{r}-\vec{r}\,')\,|\phi\rangle$$

We can verify that the action of $\hat{\pi}(\vec{r})$ is

$$\hat{\pi}(\vec{r})\,|\phi\rangle = -i\,\frac{\delta}{\delta\phi(x)}\,|\phi\rangle$$

The equation can then be written

$$\hat{\Phi}(\vec{r})\left(-i\frac{\delta}{\delta\phi(\vec{r}\,')}\right)|\phi\rangle - \left(-i\frac{\delta}{\delta\phi(\vec{r}\,')}\right)\phi(\vec{r})\,|\phi\rangle = i\delta^3(\vec{r}-\vec{r}\,')|\phi\rangle$$

In the second term, the variational derivative can act either on the factor of $\phi(\vec{r})$ or on $|\phi\rangle$. Its action on $|\phi\rangle$ cancels the first term. Its action on $\phi(\vec{r})$ is

$$\frac{\delta}{\delta\phi(\vec{r}\,')}\phi(\vec{r}) = \delta(\vec{r}-\vec{r}\,')$$

This gives a term that matches the right side of the equation.

(c) The vacuum state satisfies

$$\hat{a}_{\vec{p}}\,|0\rangle = 0 \quad \text{for all } \vec{p}$$

The projection onto a simultaneous eigenstate $|\phi\rangle$ of $\hat{\phi}(\vec{r})$ for all $\vec{r}$ is

$$\langle\phi|\,\hat{a}_{\vec{p}}\,|0\rangle = 0$$

Inserting the expression for $\hat{a}_{\vec{p}}$ in terms of $\hat{\phi}(\vec{r})$ and $\hat{\pi}(\vec{r})$, we have

$$\frac{1}{\sqrt{2\omega_p}}\int d^3r\, e^{-i\vec{p}\cdot\vec{r}}\,\langle\phi|\left(\omega_p\hat{\phi}(\vec{r}) + i\,\hat{\pi}(\vec{r})\right)|0\rangle = 0$$

Upon inserting the field representations of $\hat{\phi}(\vec{r})$ and $\hat{\pi}(\vec{r})$, this becomes

$$\int d^3r\, e^{-i\vec{p}\cdot\vec{r}} \left(\omega_p \phi(\vec{r}) + \frac{\delta}{\delta\phi(\vec{r})} \right) \langle \phi | 0 \rangle = 0$$

Then inverse Fourier transform gives a functional differential equation for $\langle \phi | 0 \rangle$:

$$\frac{\delta}{\delta\phi(\vec{r})} \langle \phi | 0 \rangle = -\int \frac{d^3p}{(2\pi)^3} e^{+i\vec{p}\cdot\vec{r}} \left(\int d^3r'\, e^{-i\vec{p}\cdot\vec{r}'} \omega_p \phi(\vec{r}') \right) \langle \phi | 0 \rangle$$

$$= -\int d^3r' \left(\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \omega_p \right) \phi(\vec{r}') \langle \phi | 0 \rangle$$

(d) The solution for the vacuum wavefunctional in Eq. (14.65) is

$$\langle \phi | 0 \rangle = \exp\left(-\frac{1}{2} \int d^3x \int d^3y\, \mathcal{E}(\vec{x},\vec{y})\, \phi(\vec{x})\, \phi(\vec{y}) \right)$$

Its variational derivative is

$$\frac{\delta}{\delta\phi(\vec{r})} \langle \phi | 0 \rangle = \exp\left(-\frac{1}{2} \int d^3x \int d^3y\, \mathcal{E}(\vec{x},\vec{y}) \phi(\vec{x}) \phi(\vec{y}) \right) \left(-\int d^3y\, \mathcal{E}(\vec{r},\vec{y}) \phi(\vec{y}) \right)$$

$$= -\int d^3r'\, \mathcal{E}(\vec{r},\vec{r}')\, \phi(\vec{r}') \langle \phi | 0 \rangle$$

The functional differential equation implies that $\mathcal{E}$ is

$$\mathcal{E}(\vec{r},\vec{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \omega_p$$

(e) The function $\mathcal{E}(\vec{x},\vec{y})$ can be written

$$\mathcal{E}(r) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{r}}\, \sqrt{p^2+m^2}$$

where $\vec{r} = \vec{x}-\vec{y}$. Using spherical coordinates for $\vec{p}$, this becomes

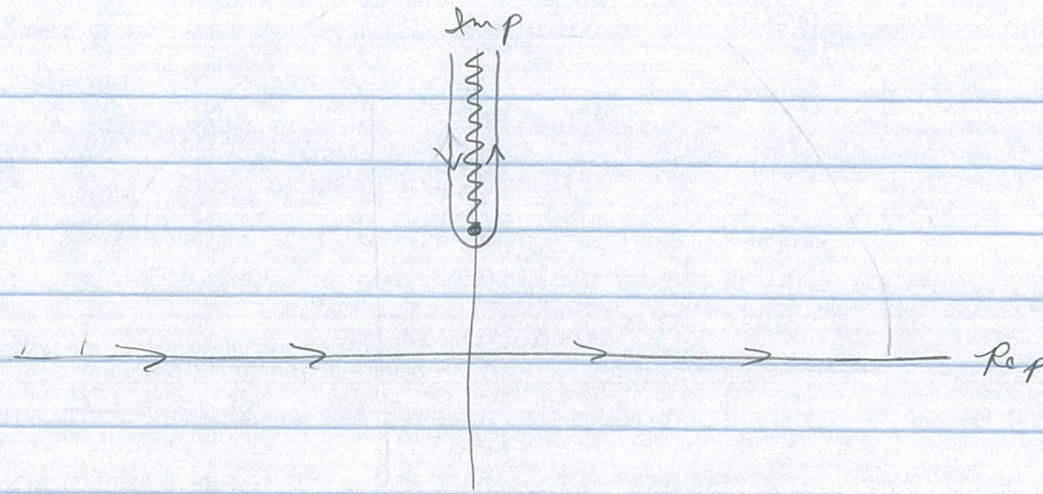
$$\mathcal{E}(r) = \frac{1}{(2\pi)^3}\, 2\pi \int_0^\infty p^2\, dp \int_{-1}^{+1} d(\cos\theta)\, e^{ipr\cos\theta} \sqrt{p^2+m^2}$$

$$= \frac{1}{4\pi^2} \int_0^\infty dp\; p^2\, \frac{1}{ipr}\left(e^{ipr} - e^{-ipr}\right) \sqrt{p^2+m^2}$$

This can be written as an integral over p from $-\infty$ to $+\infty$

$$\mathcal{E}(r) = \frac{1}{4\pi^2 i r} \int_{-\infty}^{\infty} dp\;\, p\sqrt{p^2+m^2}\, e^{ipr}$$

As a function of complex p, the integrand decreases exponentially along the positive imaginary axis and it has a square root branch cut running from $p = +im$ to $+i\infty$. If there was a convergence factor (such as $1/(p^2+\Lambda^2)$) that suppressed the integrand at large p along the real axis, we could close the contour with a semicircle at ∞ in the upper half plane. We will proceed as if there was such a convergence factor. The integration contour can then be deformed to wrap around the branch cut running from $+i\infty-\epsilon$ to im to $+i\infty+\epsilon$



Parameterize the contour by $p = imy$, where y runs from $\infty + i\epsilon$ to 1 to $\infty - i\epsilon$. The function becomes

$$\mathcal{E}(r) = \frac{1}{4\pi^2 i r}\left(\int_{\infty - i\epsilon}^{1} im\,dy\; imy\; m^2 \sqrt{-(y+i\epsilon)^2+1}\; e^{-mry} \right.$$

$$\left. + \int_1^{\infty} im\,dy\; imy\; m^2 \sqrt{-(y-i\epsilon)^2+1}\; e^{-mry} \right)$$

$$= \frac{1}{4\pi^2 i r}(-m^4)\int_1^{\infty} dy\, y \left(e^{i\pi/2}\sqrt{y^2-1} - e^{-i\pi/2}\sqrt{y^2-1} \right) e^{-mry}$$

$$= \frac{-m^4}{4\pi^2 i r}\, 2i \int_1^{\infty} dy\, y \sqrt{y^2-1}\, e^{-my}$$

$$= -\frac{m^4}{2\pi^2 r} \cdot \frac{1}{mr} K_2(mr)$$

where $K_2(z)$ is a Bessel function. Its limiting behavior at small m is

$$K_2(mr) \longrightarrow \frac{2}{(mr)^3}$$

Thus the limit of the function as $m \to 0$ is

$$\mathcal{E}(r) \longrightarrow -\frac{1}{\pi^2 r^5}$$