Fermion Path Integral

Dirac spinor field: $\psi_i(x)$, $i=1,2,3,4$

$$\bar{\psi}_i(x) = \sum_j \psi_j^\dagger(x)(\gamma_0)_{ji}$$

Free field theory

Lagrangian: $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

$$= \psi^\dagger i\partial_0\psi - \bar{\psi}(i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

canonical momentum conjugate to ψ_i:

$$\pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_i)} = i\psi_i^\dagger$$

canonical commutation relations

$$\{\psi_i(\vec{r},t), \psi_j(\vec{r}',t)\} = 0$$

$$\{\psi_i(\vec{r},t), i\psi_j^\dagger(\vec{r}',t)\} = i\hbar\,\delta^3(\vec{r}-\vec{r}')$$

$$\{i\psi_i^\dagger(\vec{r},t), i\psi_j^\dagger(\vec{r}',t)\} = 0$$

classical limit: $\hbar \to 0$

$\psi_i(\vec{r},t)$ is a Grassmann field

<u>Fermionic functional integral</u>

infinite-dimensional Grassman algebra
generated by $\psi_i(x)$, $i=1,2,3,4$ at each spacetime point x
$\psi_i^\dagger(x)$, $i=1,2,3,4$ "

Grassmann functional integral

$\int D\bar{\psi} D\psi \; F(\psi, \bar{\psi})$

invariant under shifting field $\psi(x)$
by Grassmann function $\xi(x)$ independent of ψ

$\int D\bar{\psi} D\psi \; F(\psi, \bar{\psi}) = \int D\bar{\psi} D\psi \; F(\psi + \xi, \bar{\psi})$

Gaussian functional integral

$\int D\bar{\psi} D\psi \; \exp\left(-\int d^4_x \, \bar{\psi} M \psi\right) = N \det M$

where N is infinite constant

action for fermionic field: $S[\psi, \bar{\psi}]$

vacuum expectation value of time-ordered product

$$\langle \Omega | T\, \mathcal{O} | \Omega \rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\psi,\bar{\psi}]}\, \mathcal{O}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\psi,\bar{\psi}]}}$$

generating functional for Green function

$$Z[\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\psi,\bar{\psi}] + i\int d^4x (\bar{\psi}\eta + \bar{\eta}\psi)}$$

depends on Grassmann source $\eta_i(x)$, $i = 1,2,3,4$

$$\frac{\delta}{\delta \eta_i(x)} Z[\eta] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iS + i\int d^4x (\bar{\psi}\eta + \bar{\eta}\psi)} \underbrace{\frac{\delta}{\delta \eta_i(x)}\, i\int d^4y\, \bar{\psi}_j(y)\, \eta_j(y)}$$

$$= i\int d^4y\, \bar{\psi}_j(y)\, (-1)\, \delta_{ij}\, \delta^4(x-y)$$

$$= -i\, \bar{\psi}_i(x)$$

$$\frac{\delta}{\delta \bar{\eta}_i(x)} Z[\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS + i\int d^4x (\bar{\psi}\eta + \bar{\eta}\psi)} \left(+i\, \psi_i(x)\right)$$

$$\left(-i \frac{\delta}{\delta \bar{\eta}_i(x)}\right)\left(i \frac{\delta}{\delta \eta_i(y)}\right) Z[\eta]$$

$$= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp\left(iS + i\int d^4x (\bar{\psi}\eta + \bar{\eta}\psi)\right) \psi_i(x)\, \bar{\psi}_j(y)$$

Propagator

$$\langle \Omega | T \psi_i(x) \bar{\psi}_j(y) | \Omega \rangle = \frac{1}{Z[0,0]} \frac{\delta}{\delta \bar{\eta}_i(x)} \frac{\delta}{\delta \eta_j(y)} Z[\eta] \Big|_{\eta=0\ \bar{\eta}=0}$$

Free field theory: $S[\psi, \bar{\psi}] = \int d^4x\, \bar{\psi} (i\partial\!\!\!/ - m) \psi$

generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left(i \int d^4x \left(\bar{\psi} (i\partial\!\!\!/ - m + i\epsilon) \psi + \bar{\psi}\eta + \bar{\eta}\psi \right) \right)$$

shift field $\psi \to \psi + (i\partial\!\!\!/ - m + i\epsilon)^{-1} \eta$

$\bar{\psi} \to \bar{\psi} + \bar{\eta} (i\partial\!\!\!/ - m + i\epsilon)^{-1}$

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left(i \int d^4x\, \bar{\psi} (i\partial\!\!\!/ - m + i\epsilon) \psi - i \int d^4x\, \bar{\eta} (i\partial\!\!\!/ - m + i\epsilon)^{-1} \eta \right)$$

$$= \exp\Big(\underbrace{-i \int d^4x\, \bar{\eta} (i\partial\!\!\!/ - m + i\epsilon)^{-1} \eta}_{\int d^4x \int d^4y\, \bar{\eta}_i(x) \left[\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(y-x)} \left(\frac{1}{k\!\!\!/ - m + i\epsilon} \right)_{ij} \right] \eta_j(y)} \Big) \times N \frac{1}{\det(i\partial\!\!\!/ - m + i\epsilon)}$$

$\langle 0 | T \psi_i(x) \bar{\psi}_j(y) | \Omega \rangle$

$$= \left(-i \frac{\delta}{\delta \bar{\eta}_i(x)} \right) \left(i \frac{\delta}{\delta \eta_j(y)} \right) \left(-i \int d^4z \int d^4w\, \bar{\eta}_i(x) \left[\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(y-x)} \left(\frac{1}{k\!\!\!/ - m + i\epsilon} \right)_{ij} \right] \eta_j(y) \right) \Big|_{\eta=0}$$

$$= (-1) \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(y-x)} \left(\frac{-i}{k\!\!\!/ - m + i\epsilon} \right)_{ij}$$

Feynman propagator for Dirac spinor!