Path Integral for Anharmonic Oscillator

The action for an anharmonic ocscillator is

$$S[x] = \int_{-\infty}^{+\infty} \mathrm{d}t\, L, \qquad L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - \tfrac{1}{24}\lambda x^4.$$

The complete propagator can be expressed as a ratio of path integrals:

$$\langle 0|\mathrm{T}\hat{x}(t_1)\hat{x}(t_0)|0\rangle = \frac{\int \mathrm{D}x\, e^{iS[x]}\, x(t_1)x(t_0)}{\int \mathrm{D}x\, e^{iS[x]}},$$

where the integral is over all paths $x(t)$. The path integral is invariant under a shift in the path:

$$x(t) \longrightarrow x(t) + \varepsilon(t).$$

A. What is the change δL in the Lagrangian at first order in ε?

$$\delta L = \tfrac{1}{2}\cdot 2\dot{x}\dot{\varepsilon} - \tfrac{1}{2}\omega^2 \cdot 2x\varepsilon - \tfrac{1}{24}\lambda\cdot 4x^3\varepsilon = \dot{x}\dot{\varepsilon} - \omega^2 x\varepsilon - \tfrac{1}{6}\lambda x^3\varepsilon$$

B. The first-order change in the action is $\delta S = \int \mathrm{d}t\, \delta L$. Using integration by parts, express δS as an integral over t with an explicit factor of $\varepsilon(t)$.

$$\delta S = \int dt \left(\dot{x}\dot{\varepsilon} - \omega^2 x\varepsilon - \tfrac{1}{6}\lambda x^3\varepsilon\right)$$

$$= \int dt \left(-\ddot{x} - \omega^2 x - \tfrac{1}{6}\lambda x^3\right)\varepsilon$$

C. The first-order change in $x(t_0)$ is $\delta x(t_0) = \varepsilon(t_0)$. Using a delta function, express $\delta x(t_0)$ as an integral over t with an explicit factor of $\varepsilon(t)$.

$$\delta x(t_0) = \varepsilon(t_0) = \int dt\ \delta(t-t_0)\,\varepsilon(t)$$

D. The first order change in $e^{iS}x(t_0)$ is

$$\delta\big(e^{iS}x(t_0)\big) = e^{iS}[i\delta S\, x(t_0) + \delta x(t_0)].$$

Express this as an integral over t with an explicit factor of $\varepsilon(t)$.

$$\delta\big(e^{iS}x(t_0)\big) = e^{iS}\left(i\int dt\left[-\ddot{x} - \omega^2 x - \tfrac{1}{6}\lambda x^3\right]\varepsilon \cdot x(t_0) + \int dt\ \delta(t-t_0)\,\varepsilon(t)\right)$$

$$= e^{iS}\int dt\left(i\left[-\ddot{x}(t) - \omega^2 x(t) - \tfrac{1}{6}\lambda x^3(t)\right]x(t_0) + \delta(t-t_0)\right)\varepsilon(t)$$

The first order change in $e^{iS[x]}x(t_0)$ from the shift $x(t) \to x(t) + \varepsilon(t)$ is

$$\delta\big(e^{iS[x]}x(t_0)\big) = e^{iS[x]} \int_{-\infty}^{+\infty} \mathrm{d}t \,\Big(i\big[-\ddot{x}(t) - \omega^2 x(t) - \tfrac{1}{6}\lambda x^3(t)\big]x(t_0) + \delta(t - t_0)\Big)\varepsilon(t).$$

The path integral weighted by only $x(t_0)$ is

$$\int \mathrm{D}x \, e^{iS[x]} \, x(t_0).$$

This path integral is invariant under the shift $x(t) \to x(t) + \varepsilon(t)$ for any infinitesimal function $\varepsilon(t)$.

E. Express the first-order change in this path integral as an integral over t with an explicit factor of $\varepsilon(t)$.

$$\delta\left(\int \mathrm{D}x \, e^{iS[x]} \, x(t_0)\right)$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}t \, \varepsilon(t) \times \int \mathcal{D}x \, e^{iS[x]} \Big(i\big[-\ddot{x}(t) - \omega^2 x(t) - \tfrac{1}{6}\lambda x^3(t)\big] x(t_0) + \delta(t-t_0)\Big)$$

F. If $\int dt\, f(t)\varepsilon(t) = 0$ for all functions $\varepsilon(t)$, the function $f(t)$ must be 0. Deduce from part E that a function of t involving path integrals must be zero.

$$\int \mathcal{D}x \, e^{iS[x]} \Big(i\big[-\ddot{x}(t) - \omega^2 x(t) - \tfrac{1}{6}\lambda x^3(t)\big] x(t_0) + \delta(t-t_0)$$

G. By dividing each term in this equation by the unweighted path integral, obtain the Schwinger-Dyson equation for the complete propagator.

$$\left[\left(\frac{d}{dt}\right)^2 + \omega^2\right] \langle 0|\mathrm{T}\hat{x}(t)\hat{x}(t_0)|0\rangle = -\frac{1}{6}\lambda \langle 0| T \, \hat{x}^3(t)\, \hat{x}(t_0)|0\rangle - i\,\delta(t-t_0)$$