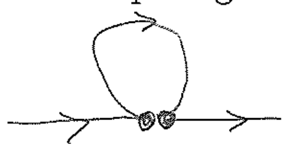
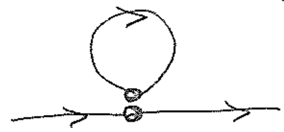
Renormalization of Dirac Spinor Field Theory

A possible Lagrangian for a self-interacting Dirac spinor field is

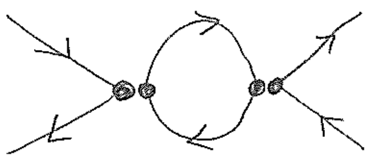
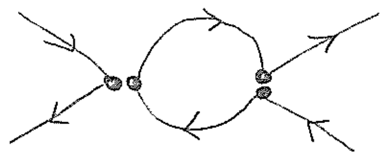
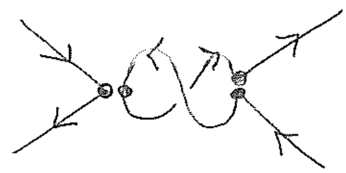
$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \tfrac{1}{2}g(\bar{\psi}\psi)^2 + \delta\mathcal{L}.$$

The interaction vertex can be represented by a double dot with two lines with consistent arrows attached to each dot. It can be expressed as $-ig\mathbb{1}\otimes\mathbb{1}$.

A. Draw the 2 one-loop diagrams for the fermion self-energy.

B. Draw 3 of the 9 one-loop diagrams for the 1PI 4-fermion Green function.

A diagram for the 1PI Green function with E external fermion lines can have P propagators, V vertices, and L loops, where these numbers satisfy the topological identities

$$L = P - V + 1, \qquad 2P + E = 4V.$$

C. Verify that the vertex satisfies the 1st topological identity.

$$L=0,\ P=0,\ V=1 \implies L = P - V + 1$$

D. Verify that if two lines in a diagram are connected by an additional vertex, the changes in the numbers satisfy the 1st topological identity: $\Delta L = \Delta P - \Delta V$.

$$\Delta L = 1,\ \Delta P = 2,\ \Delta V = 1 \implies \Delta L = \Delta P - \Delta V$$

E. Verify that both sides of the 2nd topological identity count the total number of lines attached to all the vertices in the diagram.

each vertex has 4 incoming lines
each propagator is attached to two vertices, each external line is attached to 1 vertex

The superficial degree of divergence of the diagram is $D = 4L - P$.

F. Use the 1st topological identity to eliminate L.

$$D = 4(P-V+1) - P = 3P - 4V + 4$$

G. Use the 2nd topological identity to eliminate P.

$$D = 3\cdot\tfrac{1}{2}(4V-E) - 4V + 4 = 4 - \tfrac{3}{2}L + 2V$$

The superficial degree of divergence of a diagram for the 1PI Green function with E external fermion lines is $D = 4 - \frac{3}{2}E + 2V$.

H. Verify that for any given E, there are 1PI diagrams with E external lines with $D \geq 0$, and also that there are diagrams for which D is arbitrarily large.

$D \geq 0$ if $V \geq \frac{3}{4}L - 2$ as $V \to \infty$, $D \to \infty$

Conclude that the cancellation of UV divergences requires a counterterm vertex with all possible numbers E of external legs, and also that its Feynman rule must be a polynomial of arbitrarily high order in the external momenta.

Renormalizability therefore requires the Lagrangian to include terms with all possible equal numbers of factors of ψ and $\bar{\psi}$ and all possible even numbers m of derivatives ∂ that are consistent with the symmetries.

I. Given that the action $S = \int d^4x\, \mathcal{L}$ is dimensionless and that a coordinate x^μ has mass dimension -1, deduce the mass dimension of $\mathcal{L}$.

$$[x^\mu] = \frac{1}{M} \Longrightarrow [\mathcal{L}] = M^4$$

J. Given that a derivative ∂^μ has mass dimension $+1$, use the term $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ in $\mathcal{L}$ to deduce the mass dimension of ψ.

$$[\bar{\psi}\gamma^\mu\partial_\mu\psi] = M^4, \ [\partial_\mu] = M \Longrightarrow [\psi] = M^{3/2}$$

K. Deduce the mass dimension of an operator $\mathcal{O}_{mn}$ with n factors of both ψ and $\bar{\psi}$ and m derivatives ∂.

$$[\mathcal{O}_{mn}] = [\partial]^m [\psi]^{2n} = M^{m+3n}$$

L. Given a term $G_{mn}\mathcal{O}_{mn}$ in $\mathcal{L}$, express the coupling constant G_{mn} as a product of a dimensionless coupling constant $\hat{g}_{mn}$ and the appropriate number of factors of a mass scale M.

$$G_{mn} = \frac{\hat{g}_{mn}}{M^{m+3n-4}}$$

A basis of fermion bilinears with simple Lorentz transformation properties is

$$\bar{\psi}\psi, \qquad \bar{\psi}\gamma^\mu\psi, \qquad \bar{\psi}\sigma^{\mu\nu}\psi, \qquad \bar{\psi}\gamma^\mu\gamma_5\psi, \qquad \bar{\psi}\gamma_5\psi.$$

They transform under parity with factors of $+1$, $(-1)^\mu$, $(-1)^\mu(-1)^\nu$, $-(-1)^\mu$, and -1, respectively.

M. Write down the 5 independent interaction terms in $\mathcal{L}$ of the form $\bar{\psi}\Gamma_1\psi\,\bar{\psi}\Gamma_2\psi$ allowed by Lorentz invariance and parity symmetry.

$$\bar{\psi}\psi\bar{\psi}\psi \quad \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi \quad \bar{\psi}\sigma^{\mu\nu}\psi\bar{\psi}\sigma_{\mu\nu}\psi \quad \bar{\psi}\gamma^\mu\gamma_5\psi\bar{\psi}\gamma_\mu\gamma_5\psi \quad \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi$$

N. Write down the 3 additional interaction terms if parity is not a symmetry.

$$\bar{\psi}\psi\bar{\psi}\gamma_5\psi \quad \bar{\psi}\gamma^\mu\gamma_5\psi\bar{\psi}\gamma_\mu\gamma_5\psi \quad \epsilon_{\mu\nu\lambda\sigma}\bar{\psi}\sigma^{\mu\nu}\psi\bar{\psi}\sigma^{\lambda\sigma}\psi$$

O. What is the largest value of N for which an interaction term of the form $\bar{\psi}\Gamma_1\psi\,\bar{\psi}\Gamma_2\psi \ldots \bar{\psi}\Gamma_N\psi \ldots$ is nonzero?

$N = 4$