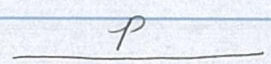
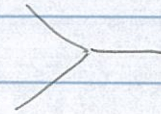
Mass Renormalization 1

real scalar field theory

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m_0^2\phi^2 - \tfrac{1}{6}g\phi^3$$

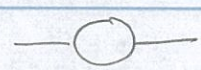
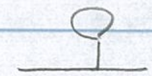
Feynman rules

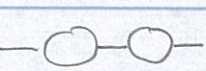
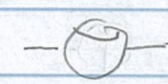
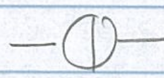


$= \dfrac{i}{p^2 - m_0^2 + i\epsilon}$  $= -ig$

complete propagator

$D(p^2) =$ 

$+ \Big($  $+$  $\Big)$ $\qquad g^2$

$+ \Big($  $+$  $+$ 

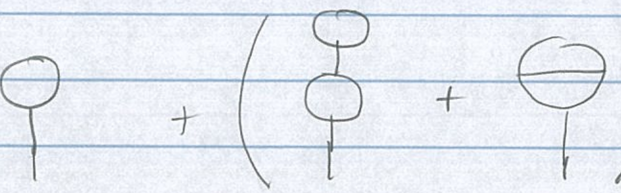
$+$ tadpole diagrams $\Big)$ $\qquad g^4$

$+ \cdots$

physical mass m_{ph} defined by pole in complete propagator

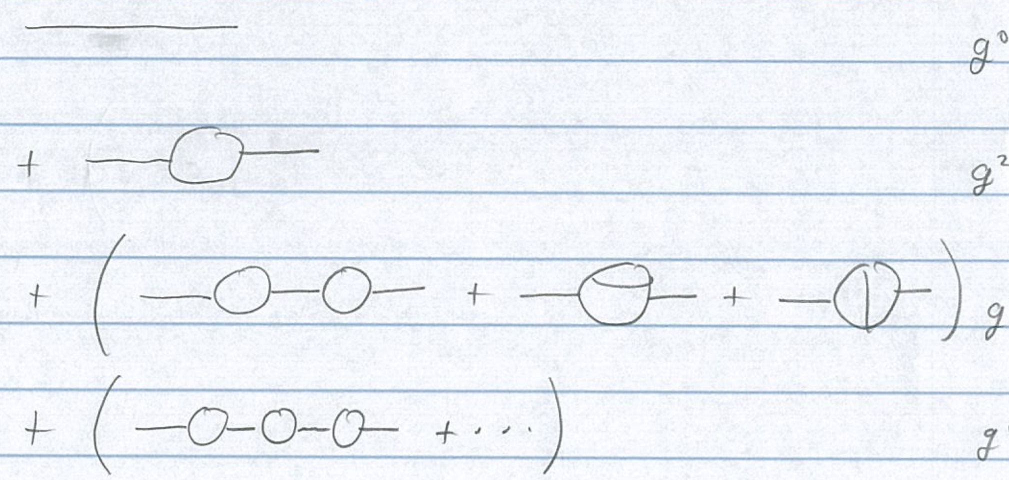
$$D(p^2) \longrightarrow \frac{iZ}{p^2 - m_{ph}^2 + i\epsilon} \quad \text{as } p^2 \longrightarrow m_{ph}^2$$

tadpole condition: $\langle \phi \rangle = 0$

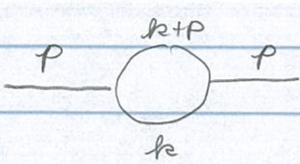
 $+ \cdots = 0$

$\Rightarrow$ can omit all diagrams with tadpoles

complete propagator (with tadpole condition)

$D(p^2) =$ 

one-loop correction

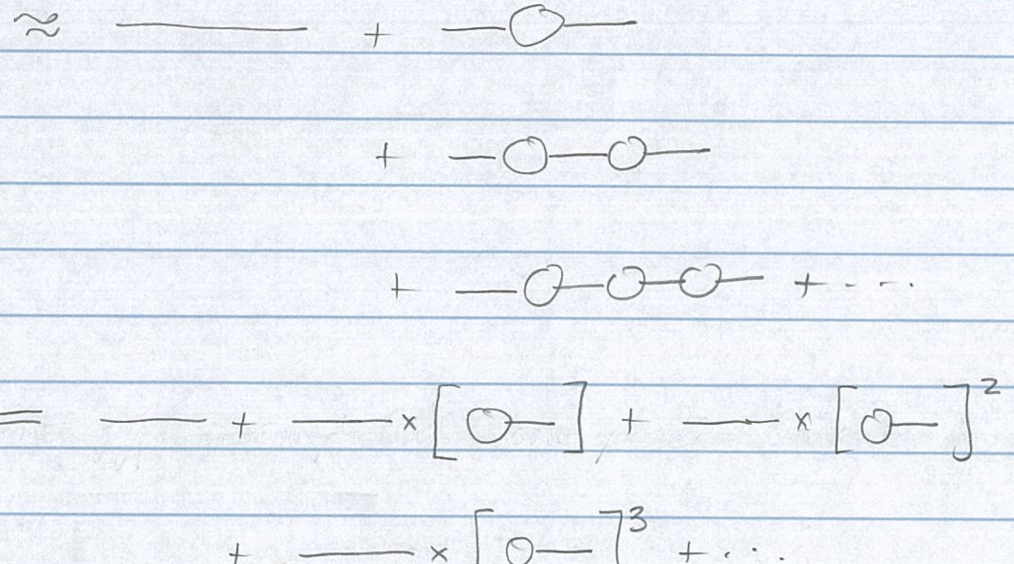


$$= \frac{i}{p^2-m_0^2+i\epsilon}\left[(-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_0^2+i\epsilon}\,\frac{i}{(k+p)^2-m_0^2+i\epsilon}\right]\frac{i}{p_0^2-m^2+i\epsilon}$$

$$= \frac{i}{p^2-m_0^2+i\epsilon}\left[\frac{1}{2}(-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_0^2+i\epsilon}\,\frac{i}{(k+p)^2-m_0^2+i\epsilon}\right]\frac{i}{p^2-m_0^2+i\epsilon}$$

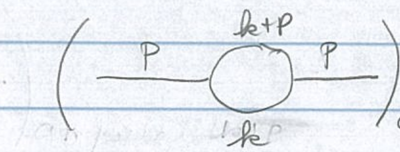
complete propagator include geometric series of these one-loop subdiagrams

$D(p^2) \approx$



self-energy : $\Pi(p^2)$

$-i\,\Pi(p^2) =$



amputated

$$= \frac{1}{2}(-ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2 - m_0^2 + i\epsilon}\, \frac{i}{(k+p)^2 - m_0^2 + i\epsilon}$$

regularize with Euclidean momentum cutoff Λ

$$= \frac{1}{2} g^2 \times \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m_0^2} - 1 - \int_0^1 dx\, \log\frac{m_0^2 - x(1-x)p^2 - i\epsilon}{m_0^2}\right]$$

Lorentz invariant function of $-p^2 - i\epsilon$

sum geometric series

$$D(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon} \sum_{n=0}^{\infty} \left[-i\Pi(p^2) \cdot \frac{i}{p^2 - m_0^2 + i\epsilon} \right]^n$$

$$= \frac{i}{p^2 - m_0^2 + i\epsilon} \frac{1}{1 - \frac{\Pi(p^2)}{p^2 - m_0^2 + i\epsilon}}$$

$$= \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}$$

$\Longrightarrow$ no double pole, but single pole is shifted

physical mass m_{ph} satisfies

$$p^2 - m_0^2 - \Pi(p^2) = 0 \quad \text{at } p^2 = m_{ph}^2$$

$$m_{ph}^2 - m_0^2 + \frac{g^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m_0^2} - 1 - \int_0^1 dx \log \frac{m_0^2 - x(1-x) m_{ph}^2 - i\epsilon}{m_{ph}^2} \right] = 0$$

solve for m_{ph}^2 to 1st order in g^2:

(can set $m_{ph}^2 = m_0^2$ in order-g^2 term)

$$m_{ph}^2 = m_0^2 - \frac{g^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m_0^2} - 1 - \underbrace{\int_0^1 dx \log(1 - x + x^2)}_{\frac{\pi}{\sqrt{3}} - 2} \right] + O(g^4)$$

$$= m_0^2 - \frac{g^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m_0^2} - \frac{\pi}{\sqrt{3}} + 1 \right] + O(g^4)$$

invert to obtain m_0^2 to 1st order in g^2

(can set $m_0^2 = m^2$ in order-g^2 term)

$$m_0^2 = m_{ph}^2 + \frac{g^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m_{ph}^2} - \frac{\pi}{\sqrt{3}} + 1\right] + O(g^4)$$

eliminate m_0 from complete propagator in favor of m_{ph}

(can set $m_0^2 = m^2$ in order-g^2 term)

$$p^2 - m_0^2 - \Pi(p^2)$$

$$= p^2 - m_0^2 + \frac{g^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m_0^2} - 1 - \int_0^1 dx \log\frac{m_0^2 - x(1-x)p^2 - i\epsilon}{m_0^2}\right] + O(g^4)$$

$$= p^2 - \left(m_{ph}^2 + \frac{g^2}{32\pi^2}\left[\cancel{\log\frac{\Lambda^2}{m_{ph}^2}} - \frac{\pi}{\sqrt{3}} + 1\right]\right)$$

$$+ \frac{g^2}{32\pi^2}\left[\cancel{\log\frac{\Lambda^2}{m_{ph}^2}} - 1 - \int_0^1 dx \log\frac{m_{ph}^2 - x(1-x)p^2 - i\epsilon}{m_{ph}^2}\right] + O(g^4)$$

$$= p^2 - m_{ph}^2 + \frac{g^2}{32\pi^2}\left[\frac{\pi}{\sqrt{3}} - 2 - \int_0^1 dx \log\frac{m_{ph}^2 - x(1-x)p^2 - i\epsilon}{m_{ph}^2}\right] + O(g^4)$$

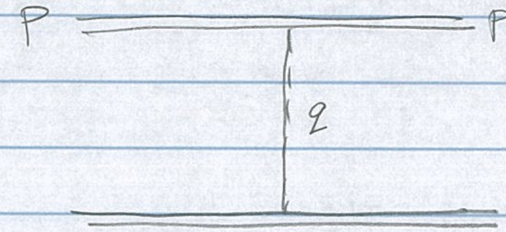
complete propagator (including self-energy through order g^2)

$$D(p^2) = \frac{i}{p^2 - m_{ph}^2 - \Pi_1(p^2)}$$

$$\Pi_1(p^2) = -\frac{g^2}{32\pi^2}\left[\frac{\pi}{\sqrt{3}} - 1 - \int_0^1 dx \frac{m_{ph}^2 - x(1-x)p^2 - i\epsilon}{m_{ph}^2}\right]$$

Potential from Scalar Exchange

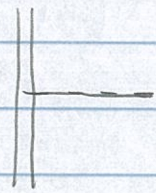
exchange of light spin-0 particle of mass m
between two heavy particles of mass M



heavy scalar field Φ
light scalar field ϕ

$$\mathcal{L} = \mathcal{L}_{heavy} + \mathcal{L}_{light} + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -\tfrac{1}{2} G \Phi^2 \phi$$



$= -iG$

scattered particles are on-shell

$$p^0 \approx M + \frac{\vec{p}^{\,2}}{2M} \qquad p'^0 \approx M + \frac{\vec{p}'^{\,2}}{2M}$$

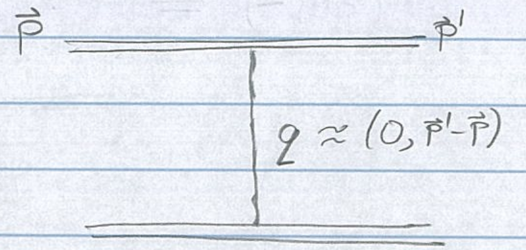
momentum transferred: $q^\mu = p'^\mu - p^\mu$

$$q^0 \approx \frac{\vec{p}'^{\,2}}{2M} - \frac{\vec{p}^{\,2}}{2M}$$

$$\vec{q} = \vec{p}' - \vec{p}$$

can approximate $q^0 \approx 0$ up to correction suppressed by $\frac{p}{M}$

matrix element



$$i\mathcal{M} = (-iG)^2 \frac{i}{q^2 - m^2 + i\epsilon}$$

$$= -iG^2 \frac{1}{q_0^2 - \vec{q}^{\,2} - m^2 + i\epsilon}$$

$$\approx iG^2 \frac{1}{\vec{q}^{\,2} + m^2}$$

$$= -i \int d^3r\, e^{-i\vec{q}\cdot r} \left(-\frac{G^2}{4\pi} \frac{e^{-mr}}{r} \right)$$

equivalent to interaction at a distance
through attractive Yukawa potential

$$V(r) = -\frac{G^2}{4\pi} \frac{e^{-mr}}{r}$$

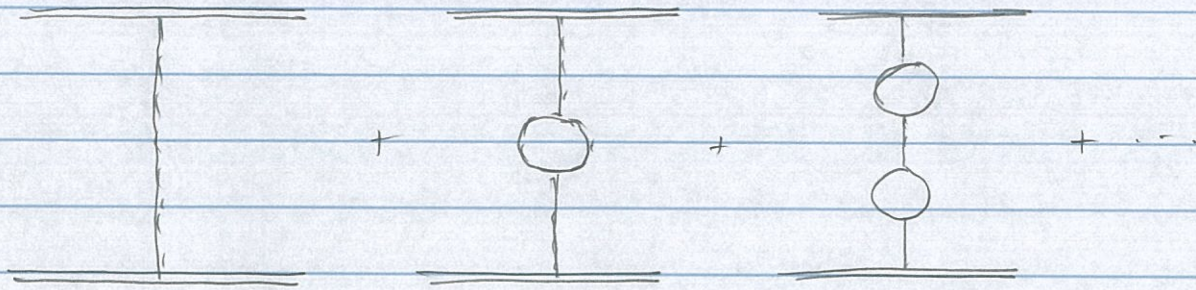
$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{q^2+m^2} = \frac{1}{8\pi^3} 2\pi \int_0^\infty q^2 dq \int_{-1}^{+1} d\cos\theta\, e^{iqr\cos\theta} \frac{1}{q^2+m^2}$$

$$= \frac{1}{4\pi^2} \int_0^\infty q^2 dq \frac{1}{iqr} \left(e^{iqr} - e^{-iqr} \right) \frac{1}{q^2+m^2}$$

$$= \frac{1}{4\pi^2 ir} \int_{-\infty}^{\infty} dq\, e^{iqr} \frac{q}{q^2+m^2}$$

$$= \frac{1}{4\pi^2 ir} 2\pi i\, e^{iqr} \frac{q}{q+im}\bigg|_{q=im} = \frac{1}{4\pi} \frac{e^{-mr}}{r}$$

effect of self-interaction of light scalar



$$\frac{1}{\vec{q}^2+m^2} \longrightarrow \frac{1}{\vec{q}^2+m_{ph}^2+\Pi_1(-\vec{q}^2)}$$

self-energy

$$\Pi_1(-\vec{q}^2) = -\frac{g^2}{32\pi^2}\left[\frac{\pi}{\sqrt{3}} - 2 - \int_0^1 dx \log\frac{m_{ph}^2 + x(1-x)\vec{q}^2}{m_{ph}^2}\right]$$

$$\approx +\frac{g^2}{32\pi^2}\left[2-\frac{\pi}{\sqrt{3}}\right] \qquad \vec{q}^2 \ll m^2$$

$$\approx \frac{g^2}{32\pi^2}\left[\log\frac{\vec{q}^2}{m_{ph}} + 4 - \frac{\pi}{\sqrt{3}}\right] \qquad \vec{q}^2 \gg m^2$$

effective potential:

$$V_{eff}(r) = -G^2\int\frac{d^3q}{(2\pi)^3}\frac{1}{\vec{q}^2+m^2+\Pi_1(-\vec{q}^2)}$$