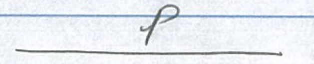
Coupling Constant Renormalization

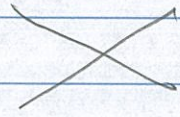
real scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{24}\lambda_0\phi^4$$

Feynman rules



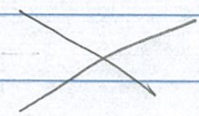
$= \dfrac{i}{p^2 - m_0^2 + i\epsilon}$


$= -i\lambda_0$

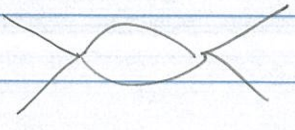
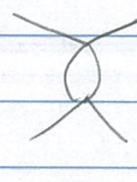
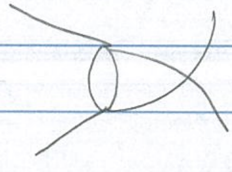
$2 \to 2$ scattering

cross section: $d\sigma = \dfrac{1}{4E_{cm}\,p_*}\,|\mathcal{M}(s,t)|^2\,\dfrac{1}{32\pi^2}\,\dfrac{p_*}{E_{cm}}\,d\Omega_*$

1^{st} order in λ_0

$i\mathcal{M}_1 =$  $= -i\lambda$

2^{nd} order in λ_0

$i\mathcal{M}_2 =$  $+$  $+$ 

diagrams are ultraviolet divergent integrals

impose Lorentz invariant momentum cutoff Λ

$$i\mathcal{M}_2(s,t) = i\frac{\lambda_0^2}{32\pi^2}\left[f_0(s) + f_0(t) + f_0(u) \right]$$

$$f_0(s) = \left[\log\frac{\Lambda^2}{m^2} - 1 - \int_0^1 dx\,\log\frac{-x(1-x)s+m^2-i\epsilon}{m^2}\right]$$

diverges as $\Lambda \to \infty$

<u>possible interpretations</u>

- ϕ^4 interaction is unphysical?
- perturbation theory in λ_0 not valid?
- QFT intrinsically singular at short distance
  divergence as $\Lambda \to \infty$
  $\Longrightarrow$ sensitivity to short distances

<u>Problem</u>

relation between cross section σ
and parameter λ_0 in $\mathcal{L}$
is sensitive to short distances

<u>Solution</u>

express σ in terms of coupling constant λ_r
defined in terms of physical observables

define "physical" coupling constant λ_{ph}
in terms of low-energy $2 \to 2$ scattering

scattering at threshold: $s \to 4m^2$ ($\Rightarrow t \to 0, u \to 0$)

$$\mathcal{M} \longrightarrow -\lambda_{ph} \quad \text{(to all orders in } \lambda_{ph}\text{)}$$

Express $\mathcal{M}(s,t)$ as expansion in powers of λ_{ph}
instead of λ_0

- calculate λ_{ph} as expansion in powers of λ_0

to order λ_0^2,
$$\lambda_{ph} = \lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left[\left(\log\frac{\Lambda^2}{m^2} - 3\right) + 2\left(\log\frac{\Lambda^2}{m^2} - 1\right)\right]$$
$$= \lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left[3\log\frac{\Lambda^2}{m^2} - 5\right]$$

- invert to obtain λ_0 as expansion in powers of λ_{ph}
(can replace λ_0^2 by λ_{ph}^2 at this order)

$$\lambda_0 = \lambda_{ph} + \frac{\lambda_{ph}^2}{32\pi^2}\left[3\log\frac{\Lambda^2}{m^2} - 5\right] + O(\lambda_{ph}^3)$$

- eliminate λ_0 from $\mathcal{M}$ in favor of λ_{ph}
(can replace λ_0^2 by λ_{ph}^2 at this order

At higher orders in perturbation,

also need mass renormalization

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{24}\lambda_0^2\phi^4$$

m_0 is not equal to physical mass m of particle

calculate m, λ_r as power series in λ_0

eliminate m_0, λ_0 from cross section
in favor of m, λ_r

expand cross section in powers of λ_r

UV divergences cancel at each order in λ_r

QFT cannot predict cross section $\sigma(s,t)$
as function of m_0, λ_0

but it can predict $\sigma(s,t)$ as function of m, λ_r

$$\mathcal{M}(s,t) = -\left[\lambda_{ph} + \frac{\lambda_{ph}^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 5\right)\right]$$

$$+ \frac{\lambda_{ph}^2}{32\pi^2}\left[f_0(s) + f_0(t) + f_0(u)\right] + O(\lambda_{ph}^3)$$

$$f_0(s) = \log\frac{\Lambda^2}{m^2} - 1 - \int_0^1 dx \, \frac{m^2 - x(1-x)s - i\epsilon}{m^2}$$

$$= -\lambda_{ph} + \frac{\lambda_{ph}^2}{32\pi^2}\left[f(s) + f(t) + f(u)\right]$$

$$f(s) = +\frac{2}{3} - \int_0^1 dx \, \frac{m^2 - x(1-x)s - i\epsilon}{m^2}$$

$\mathcal{M}(s,t)$ has no dependence on Λ
when expressed in terms of λ_{ph}

At higher orders in perturbation theory,
also need mass renormalization

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{24}\lambda_0\phi^4$$

mass parameter m_0 is not equal to physical mass m_{ph}

calculate m_{ph}, and λ_{ph} as expansions in powers of λ_0
invert to obtain m_0, λ_0 as expansions in powers of λ_{ph}
eliminate m_0, λ_0 from $\mathcal{M}$ in favor of m_{ph}, λ_{ph}
and expand in powers of λ_{ph}

$\mathcal{M}$ has no dependence on Λ at any order in λ_{ph}