Answers for José and Saletan, Chapter 8

Problem 8.3.

The products of the rotation matrices are

$$\begin{aligned} R(\theta\hat{y})\,R(\theta\hat{z}) &= \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta & \sin\theta \\ \sin\theta & \cos\theta & 0 \\ -\sin\theta\cos\theta & \sin^2\theta & \cos\theta \end{pmatrix}, \\ R(\theta\hat{z})\,R(\theta\hat{y}) &= \begin{pmatrix} \cos^2\theta & -\sin\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos\theta & \sin^2\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \end{aligned}$$

For $\theta = \frac{1}{2}\pi$, they reduce to rotations by the angle $\frac{2}{3}\pi$ around the axes $\hat{n}_1$ and $\hat{n}_2$ given by

$$\hat{n}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \hat{n}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Problem 8.6.

straightforward with use of the identity

$$\epsilon_{ijm}\epsilon_{ikn} = \delta_{jk}\delta_{mn} - \delta_{jn}\delta_{km}$$

Problem 8.8.

The inverted equations for the time derivatives of the angles are

$$\begin{aligned} \dot{\theta} &= \frac{1}{I_1}p_\theta, \\ \dot{\phi} &= \frac{1}{I_1\sin^2\theta}\left(p_\phi - \cos\theta\,p_\psi\right), \\ \dot{\psi} &= \frac{1}{I_3}p_\psi - \frac{\cos\theta}{I_1\sin^2\theta}\left(p_\phi - \cos\theta\,p_\psi\right). \end{aligned}$$

Problem 8.15.

The expression for the quadrupole tensor in terms of the inertia tensor is

$$Q_{ij} = \left(-3\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}\right)I_{kl}.$$

Problem 8.16.

The principal moments of inertia are

$$\begin{aligned} I_1 &= I_2 = \frac{4\pi}{15}\mu(R-r)(2R^4+2rR^3+2r^2R^2-3r^3R-7r^4), \\ I_3 &= \frac{8\pi}{15}\mu(R-r)(R^4+rR^3+r^2R^2+r^3R+r^4), \end{aligned}$$

where μ is the mass density.

Problem 8.17.

(a) The angular velocity vector, kinetic energy, and angular momentum vector of the cone are

$$\begin{aligned} \vec{\omega} &= \frac{2\pi h}{PR}\left[\cos(2\pi t/P)\hat{x}+\sin(2\pi t/P)\hat{y}\right], \\ T &= \frac{3\pi^2 M(6h^2+R^2)h^2}{10P(h^2+R^2)}, \\ \vec{J} &= \frac{3\pi Mh}{10P(h^2+R^2)}\left[(6h^2+R^2)R(\cos(2\pi t/P)\hat{x}+\sin(2\pi t/P)\hat{y})+(4h^2-R^2)h\hat{z}\right]. \end{aligned}$$

(b) The total torque on the cone is the derivative of the angular momentum vector:

$$\vec{N} = \frac{3\pi Mh(6h^2+R^2)R}{P(h^2+R^2)}\left[-\sin(2\pi t/P)\hat{x}+\cos(2\pi t/P)\hat{y}\right].$$

(c) The minimum coefficient of static friction for which the cone will not slip is

$$\mu_{\rm min} =$$

(d) The minimum period of rotation for which the cone will not tilt is

$$\mu_{\rm min} =$$

Problem 8.19.

(a) In the inertial frame, the angular velocity vector, the inertia tensor, the angular momentum vector are

$$\begin{aligned} \vec{\omega} &= \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \\ I &= \tfrac{1}{2}mb^2 \begin{pmatrix} \sin^2\theta\sin^2(\omega t)+\cos^2\theta & -\sin^2\theta\cos(\omega t)\sin(\omega t) & -\sin\theta\cos\theta\cos(\omega t) \\ -\sin^2\theta\cos(\omega t)\sin(\omega t) & \sin^2\theta\cos^2(\omega t)+\cos^2\theta & -\sin\theta\cos\theta\sin(\omega t) \\ -\sin\theta\cos\theta\cos(\omega t) & -\sin\theta\cos\theta\sin(\omega t) & \sin^2\theta \end{pmatrix}, \\ \vec{J} &= \tfrac{1}{2}mb^2\omega\sin\theta \begin{pmatrix} -\cos\theta\cos(\omega t) \\ -\cos\theta\sin(\omega t) \\ \sin\theta \end{pmatrix}. \end{aligned}$$

In the body-fixed frame, the angular velocity vector and the inertia tensor are

$$\begin{aligned}\vec{\omega} &= \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \\ I &= \tfrac{1}{2}mb^2 \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix}.\end{aligned}$$

(b) The torque is obtained by differentiating the angular momentum vector:

$$\vec{N} = \tfrac{1}{2}mb^2\omega^2 \sin\theta\cos\theta \begin{pmatrix} \sin(\omega t) \\ -\cos(\omega t) \\ 0 \end{pmatrix}.$$

(c) If the constraint on θ was suddenly turned off at time $t = 0$, the angle $\theta(t)$ would evolve according to the equation of motion

$$\ddot{\theta} = \omega^2 \sin\theta\cos\theta$$

with the initial conditions that $\theta(t = 0)$ and $\dot{\theta}(t = 0)$ are θ and 0, respectively.

Problem 8.20.

(a) I see no reason why the nutation rate $\dot{\theta}$ must be zero. However I will proceed to assume that it is zero. In this case, θ must have a constant value θ_0. In the limit of a small angle θ_0, $p_\phi - p_\psi$ must scale like θ_0^2. Since the disk is thin, the gravitational term in the Hamiltonian can be neglected. The expression for θ_0 then reduces to

$$\theta_0^2 = \frac{2(p_\phi - p_\psi)}{p_\psi}.$$

(b) The rates of wobble ($\dot{\phi}$) and spin ($\dot{\psi}$) depend only on p_ϕ, p_ψ, and θ_0 and are therefore constant:

$$\begin{aligned}\dot{\phi} &= \frac{1}{I_1 \sin^2\theta_0}\left(p_\phi - p_\psi \cos\theta_0\right), \\ \dot{\psi} &= \frac{1}{I_3}p_\psi - \dot{\phi}\cos\theta_0.\end{aligned}$$

(c) Applying small angle approximations and using $I_3 = 2I_1$, the expressions for $\dot{\phi}$ and $\dot{\psi}$ reduce to

$$\begin{aligned}\dot{\phi} &= \frac{p_\psi}{I_1}, \\ \dot{\psi} &= -\frac{p_\psi}{2I_1}.\end{aligned}$$

Since $|\dot{\phi}| = 2|\dot{\psi}|$, the rates of precession (wobble) and spin are in the ratio 2:1.