Answers for José and Saletan, Chapter 5

Problem 1.

The Hamiltonian is

$$H = \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + mgr\cos\theta.$$

Hamilton's equations are

$$\begin{aligned}
\dot\theta &= \frac{p_\theta}{mr^2} \\
\dot\phi &= \frac{p_\phi}{mr^2\sin^2\theta}, \\
\dot p_\theta &= \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} + mgr\sin\theta, \\
\dot p_\phi &= 0.
\end{aligned}$$

The total energy of the particle is

$$E = H + \tfrac{1}{2}m\dot r^2.$$

Thus it is not equal to the Hamiltonian H. Since H depends explicitly on the time t through $r(t)$, it is not conserved. The energy E is also not conserved, because H depends only on $r(t)$ while E depends also on $\dot r(t)$.

Problem 2.

(a) After differentiating Hamilton's equation for $\dot q$ with respect to t, it can be reduced to

$$\ddot q = -\frac{1}{2a}\dot q^2.$$

The solutions for $q(t)$ and $p(t)$ with boundary conditions $q(0) = 0$ and $p(0) = mv$ are

$$\begin{aligned}
q(t) &= 2a\log(1 + vt/(2a)), \\
p(t) &= mv(1 + vt/(2a)),
\end{aligned}$$

The limits as $t \to \infty$ are $q(t) \to \infty$, $p(t) \to \infty$, and $\dot q(t) \to 0$. The relation between the kinetic energy $T = \frac{1}{2}m\dot q^2$ and the Hamiltonian H is

$$H = Te^{-q/a}.$$

The rate at which T is dissipated is given by

$$\dot T = -\frac{m}{2a}\dot q^2.$$

(b) The Hamiltonian for an additional constant force f is.

$$H = \frac{p^2}{2m}e^{-q/a} - fae^{+q/a}.$$

Problem 4.

The Hamiltonian is

$$H = \frac{p_r^2}{2(1-2/r)} + \frac{p_\phi^2}{2r^2} + \frac{1}{r}.$$

The angular momentum $p_\phi = r^2\dot{\phi}$ is conserved. The Hamiltonian H is also conserved, because it does not depend explicitly on time. Setting $p_r = (1-2/r)\dot{r}$ and solving for $\dot{r}$, we get a first-order differential equation for $r(t)$:

$$\dot{r} = \pm\sqrt{\frac{2}{1-2/r}\left(H - \frac{p_\theta^2}{2r^2} - \frac{1}{r}\right)}.$$