Answers for José and Saletan, Chapter 2

Problem 9.

(a) Cartesian coordinates:

$$\begin{aligned} x_1 &= \ell_1 \sin\theta_1 \\ z_1 &= -\ell_1 \cos\theta_1 \\ x_2 &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \\ z_2 &= -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 \end{aligned}$$

configuration space: $\{(\theta_1, \theta_2)\colon\ -\pi < \theta_1 \le +\pi,\ -\pi < \theta_2 \le +\pi\}$
tangent bundle: $\{(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)\colon\ -\pi < \theta_1 \le +\pi,\ -\pi < \theta_2 \le +\pi,\ -\infty < \dot\theta_1 < +\infty,\ -\infty < \dot\theta_2 < +\infty\}$

(b) Lagrangian:

$$\begin{aligned} L &= \tfrac{1}{2}(m_1+m_2)\ell_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2\ell_2^2\dot\theta_2^2 + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\,\dot\theta_1\dot\theta_2 \\ &\quad +(m_1+m_2)g\ell_1\cos\theta_1 + m_2 g\ell_2\cos\theta_2 \end{aligned}$$

(c) Lagrange's equations:

$$(m_1+m_2)\ell_1\,\ddot\theta_1 + m_2\ell_2[\cos(\theta_2-\theta_1)\,\ddot\theta_2 + \sin(\theta_1-\theta_2)\,\dot\theta_2^2] + (m_1+m_2)g\sin\theta_1 = 0$$
$$m_2\ell_2\,\ddot\theta_2 + m_2\ell_1[\cos(\theta_1-\theta_2)\,\ddot\theta_1 - \sin(\theta_1-\theta_2)\,\dot\theta_1^2] + m_2 g\sin\theta_2 = 0$$

Problem 10.

Cartesian coordinates:

$$\begin{aligned} x &= (\ell+s)\sin\theta \\ z &= -(\ell+s)\cos\theta \end{aligned}$$

Lagrangian:

$$L = \tfrac{1}{2}m(\ell+s)^2\dot\theta^2 + \tfrac{1}{2}m\dot s^2 + mg(\ell+s)\cos\theta - \tfrac{1}{2}ks^2$$

Lagrange's equations:

$$(\ell+s)\ddot\theta + 2\dot s\ddot\theta = -g\sin\theta$$
$$m\ddot s = m(\ell+s)\dot\theta^2 + mg\cos\theta - ks$$

Problem 11.

(a) Cartesian coordinates:

$$\begin{aligned} x_1 &= \rho\cos(\omega t) \\ x_2 &= \rho\sin(\omega t) \\ y &= A\rho^n \end{aligned}$$

Lagrange's equation:

$$(1 + n^2A^2\rho^{2n-2})\ddot{\rho}) + (n-1)n^2A^2\rho^{2n-3}\dot{\rho}^2 = \omega^2\rho - ngA\rho^{n-1}$$

equilibrium radius for $n > 2$:

$$\rho_0 = \left(\frac{\omega^2}{ngA}\right)^{1/(n-2)}$$

equilibrium height for $n > 2$:

$$y_0 = A\left(\frac{\omega^2}{ngA}\right)^{n/(n-2)}$$

equilibrium radius and height for $n = 2$: $\rho_0 = 0$, $y_0 = 0$

(b) small vibrations: $\rho(t) = \rho_0 + \delta\rho(t)$

$$\begin{aligned} (1 + n^2A^2\rho_0^{2n-2})\delta\ddot{\rho} &= [\omega^2 - (n-1)ngA\rho_0^{n-2}]\delta\rho \\ &= -(n-2)\omega^2\delta\rho \end{aligned}$$

vibration frequency ν:

$$2\pi\nu = \sqrt{\frac{n-2}{1 + n^2A^2\rho_0^{2n-2}}}\,\omega$$

Problem 13.

The center of mass moves with velocity $[m_1/(m_1+m_2)]v$.
The spring oscilates about its equilibrium length with angular frequency $\sqrt{k/m}$.

maximum and minimum separations:

$$x_\pm = \ell \pm \sqrt{\frac{m_1m_2}{(m_1+m_2)k}}\,v$$

Problem 15.

The period T and the semimajor axis a are related by

$$T^2 = \frac{4\pi^2}{G(M+m)} a^3.$$

Problem 17.

(a) Calculate $d\vec{A}/dt$ and then use Newton's equation.

(b) Calculate the dot products

$$\begin{aligned} \vec{A} \cdot \vec{L} &= 0, \\ \vec{A} \cdot \vec{x} &= \vec{L}^2 - \alpha\mu r, \end{aligned}$$

where $\vec{L} = \vec{x} \times \vec{p}$. Compare the second equation with Eq. (2.71) for the orbit.