

Thornton and Marion

6-3

Given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) , a path between them is $(x(s), y(s), z(s))$, where the 3 functions satisfy the boundary condition

$$\begin{aligned} x(0) &= x_1 & y(0) &= y_1 & z(0) &= z_1 \\ x(1) &= x_2 & y(1) &= y_2 & z(1) &= z_2 \end{aligned}$$

The differential arclength along the path is

$$\sqrt{(dx(s))^2 + (dy(s))^2 + (dz(s))^2} = \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2} ds$$

The length of the path is the functional

$$D[x, y, z] = \int_0^1 \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2} ds$$

The path that minimizes the distance functional must satisfy the Euler equations:

$$\frac{d}{ds} \frac{x'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = 0$$

$$\frac{d}{ds} \frac{y'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = 0$$

$$\frac{d}{ds} \frac{z'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = 0$$

The quantities inside the derivative must be constants independent of s . The constants are constrained by the identity

$$\left(\frac{x'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} \right)^2 + \left(\frac{y'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} \right)^2 + \left(\frac{z'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} \right)^2 = 1$$

A choice of constants consistent with the identity is

$$\frac{x'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\frac{y'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\frac{z'(s)}{\sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}} = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

These equations imply

$$x'(s) = a f(s)$$

$$y'(s) = b f(s)$$

$$z'(s) = c f(s)$$

where $f(s)$ is some function of s . Integrating from $s=0$ to $s=1$, we get

$$x_2 - x_1 = a \int_0^1 ds f(s)$$

$$y_2 - y_1 = b \int_0^1 ds f(s)$$

$$z_2 - z_1 = c \int_0^1 ds f(s)$$

These equations determine a, b, and c.

$$a = \frac{x_2 - x_1}{\int_0^1 ds f(s)}$$

$$b = \frac{y_2 - y_1}{\int_0^1 ds f(s)}$$

$$c = \frac{z_2 - z_1}{\int_0^1 ds f(s)}$$

Integrating from 0 to s, we get

$$x(s) = x_1 + (x_2 - x_1) g(s)$$

$$y(s) = y_1 + (y_2 - y_1) g(s)$$

$$z(s) = z_1 + (z_2 - z_1) g(s)$$

where the function g(s) is given by

$$g(s) = \frac{\int_0^s ds' f(s')}{\int_0^1 ds f(s)}$$

These are equations for a straight-line path

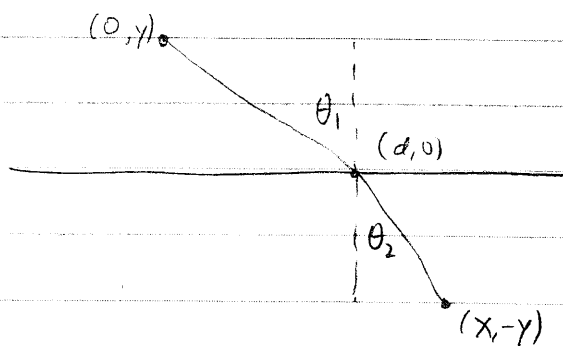
Thornton and Marwin

6-7

Light in a medium with index of refraction n moves with speed c/n

Let the interface between two media with indices of refraction n_1 and n_2 lie on the x -axis.

Consider a beam of light that passes through the points $(0, y)$ and $(x, -y)$, crossing the interface at the point $(d, 0)$.



The travel time between the points $(0, y)$ and $(x, -y)$ is

$$t = \frac{\sqrt{d^2 + y^2}}{c/n_1} + \frac{\sqrt{(x-d)^2 + y^2}}{c/n_2}$$

The minimum travel time is obtained for the value of d such that the derivative with respect to d is zero:

$$0 = \frac{n_1}{c} \frac{d}{\sqrt{d^2 + y^2}} + \frac{n_2}{c} \frac{d-x}{\sqrt{(d-x)^2 + y^2}}$$

This can be written $n_1 \sin \theta_1 = n_2 \sin \theta_2$, which is the Law of Refraction.

Thornton and Marion

6-15

(a). The functional is

$$I[y] = \int_0^1 dx (y'(x)^2 - y^2(x))$$

The Euler equation is

$$\frac{d}{dx} (2y'(x)) - (-2y(x)) = 0$$

$$y''(x) + y(x) = 0$$

The general solution is

$$y(x) = A \sin x + B \cos x$$

The solution satisfying the boundary conditions $y(0) = 0$, $y(1) = 1$ is

$$y_0(x) = \frac{\sin x}{\sin 1}$$

(b) The first derivative is

$$y_0'(x) = \frac{\cos x}{\sin 1}$$

The value of the functional for this function is

$$\begin{aligned}
 I[y_0] &= \int_0^1 dx \left[\left(\frac{\cos x}{\sin 1} \right)^2 - \left(\frac{\sin x}{\sin 1} \right)^2 \right] \\
 &= \frac{1}{\sin^2 1} \int_0^1 dx [\cos^2 x - \sin^2 x] \\
 &= \frac{1}{\sin^2 1} \cos(1) \sin(1) \\
 &= \frac{\cos(1)}{\sin(1)} \approx 0.643
 \end{aligned}$$

(c). The general equation for a straight line is

$$y(x) = Ax + B$$

The straight line satisfying the boundary conditions $y(0) = 0$ and $y(1) = 1$ is

$$y_1(x) = x$$

The value of the functional at this function is

$$\begin{aligned}
 I[y_1] &= \int_0^1 dx [1^2 - x^2] \\
 &= 1 - \frac{1}{3} = \frac{2}{3}
 \end{aligned}$$

This is larger than $I[y_0] \approx 0.643$.

Thornton and Marion

6-17

(a) The equation for the ellipse is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

A rectangle with corners at (x, y) , $(x, -y)$, $(-x, y)$, and $(-x, -y)$ will have area $(2x)(2y) = 4xy$.

To minimize the area $4xy$ subject to the constraint that (x, y) lies on the ellipse, we introduce a Lagrange multiplier λ . We look for a stationary point of the function

$$f(x, y, \lambda) = 4xy - \lambda \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 \right]$$

We set the partial derivatives with respect to x, y , and λ equal to 0:

$$4y - \lambda \frac{2x}{a^2} = 0$$

$$4x - \lambda \frac{2y}{b^2} = 0$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0$$

We can solve the first two equations for λ :

$$(4y)(4x) = \left(\lambda \frac{2x}{a^2}\right) \left(\lambda \frac{2y}{b^2}\right)$$

$$16xy = \frac{4\lambda^2}{a^2b^2} xy$$

$$\lambda = 2ab$$

We next solve the first equation for y as a function of x :

$$y = \frac{b}{a} x$$

We can then solve the 3rd equation for x , and use it to get y :

$$x = \frac{a}{\sqrt{2}} \quad y = \frac{b}{\sqrt{2}}$$

The corners of the rectangle are at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right)$.

(b). The area of the rectangle is $4xy = 2ab$

The area of the ellipse is πab .

Thus the fraction of the area of the ellipse covered by the rectangle is $\frac{2}{\pi}$.