Spring 2012

Physics 664 2nd Midterm Exam NAME ____________________

Problem 1.

A rigid body that is rotating freely has a symmetry axis $\vec{e}_3(t)$ and orthogonal principal axes of inertia $\vec{e}_1(t)$ and $\vec{e}_2(t)$ that change with time. The instantaneous angular frequency vector can be expressed as $\vec{\omega} = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$. Euler's equations for the components $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are

$$\begin{aligned} 0 &= I_\perp\dot{\omega}_1 + (I_\parallel - I_\perp)\omega_3\omega_2, \\ 0 &= I_\perp\dot{\omega}_2 - (I_\parallel - I_\perp)\omega_3\omega_1, \\ 0 &= I_\perp\dot{\omega}_3, \end{aligned}$$

where $I_\parallel$ and $I_\perp$ are the principal moments of inertia.

(A) Show that ω_3 and $\omega_1^2 + \omega_2^2 \equiv \omega_\perp^2$ are constants of the motion.

$\dot{\omega}_3 = 0 \implies \omega_3$ is a constant of the motion

$$\begin{aligned} \frac{d}{dt}(\omega_1^2+\omega_2^2) &= 2\omega_1\dot{\omega}_1 + 2\omega_2\dot{\omega}_2 \\ &= 2\omega_1\left(-\frac{I_\parallel - I_\perp}{I_\perp}\omega_3\omega_2\right) + 2\omega_2\left(\frac{I_\parallel - I_\perp}{I_\perp}\omega_3\omega_1\right) \\ &= 0 \end{aligned}$$

$\implies \omega_1^2+\omega_2^2$ is a constant of the motion

(B) Let $z(t) = \omega_1(t) + i\omega_2(t)$. Show that $z(t)$ satisfies a first-order homogeneous differential equation. Write down the most general complex solution $z(t)$ to the differential equation.

$$\left[I_\perp \dot{\omega}_1 + (I_\parallel - I_\perp)\omega_3\omega_2\right] + i\left[I_\perp\dot{\omega}_2 - (I_\parallel - I_\perp)\omega_3\omega_1\right] = 0$$

$$I_\perp(\dot{\omega}_1 + i\dot{\omega}_2) + (I_\parallel - I_\perp)\omega_3(\omega_2 - i\omega_1) = 0$$

$$I_\perp(\dot{\omega}_1 + i\dot{\omega}_2) - i(I_\parallel - I_\perp)\omega_3(\omega_1 + i\omega_2) = 0$$

$$I_\perp\dot{z} - i(I_\parallel - I_\perp)\omega_3 z = 0$$

This is a first-order homogeneous differential equation

The most general solution is

$z(t) = A e^{i\Omega t}$ where $\Omega = \frac{I_\parallel - I_\perp}{I_\perp}\omega_3$

and A is an arbitrary complex constant

(C) Find a simple solution to Euler's equations for $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ that corresponds to rotation about the principal axis $\vec{e}_1$.

$\omega_1(t) = \omega_0$ where ω_0 is a constant

$\omega_2(t) = 0$

$\omega_3(t) = 0$

(D) Find the most general solution to Euler's equations for $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$, which depends on three arbitrary real constants.

If $A = \omega_\perp e^{i\phi}$, the real and imaginary parts of $z(t)$ are

$\omega_1(t) = \omega_\perp \cos(\Omega t + \phi)$

$\omega_2(t) = \omega_\perp \sin(\Omega t + \phi)$

The general solution for $\omega_3(t)$ is a constant

$\omega_3(t) = \omega_0$

The 3 arbitrary real constants are $\omega_\perp$, ϕ, and ω_0.

(E) Suppose the solutions $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ to Euler's equations are known and that the initial principal axes $\vec{e}_1(0)$, $\vec{e}_2(0)$, and $\vec{e}_3(0)$ are given. Write down the additional differential equations that determine the principal axes $\vec{e}_1(t)$, $\vec{e}_2(t)$, and $\vec{e}_3(t)$ at later times t.

$$\frac{d}{dt}\vec{e}_1 = \vec{\omega} \times \vec{e}_1$$

$$\frac{d}{dt}\vec{e}_2 = \vec{\omega} \times \vec{e}_2$$

$$\frac{d}{dt}\vec{e}_3 = \vec{\omega} \times \vec{e}_3$$

where $\vec{\omega} = \omega_1(t)\vec{e}_1(t) + \omega_2(t)\vec{e}_2(t) + \omega_3(t)\vec{e}_3(t)$

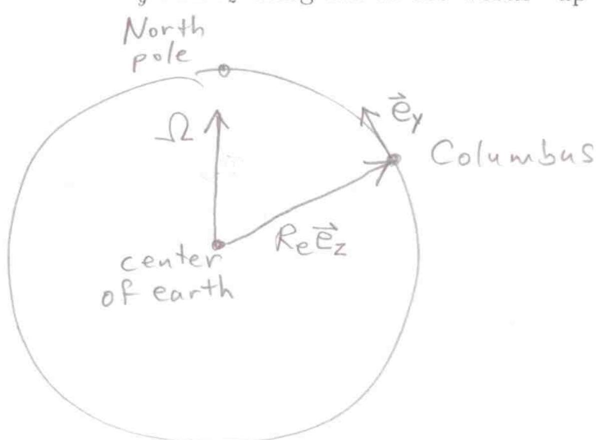
Problem 2.

In a rotating reference frame attached to the surface of the earth in Columbus, a particle has position $\vec{r}(t)$, velocity $\dot{\vec{r}}(t)$, and acceleration $\ddot{\vec{r}}(t)$. Its acceleration $\vec{a}$ in an inertial frame is

$$\vec{a} = \ddot{\vec{r}}(t) + 2\vec{\Omega} \times \dot{\vec{r}} + \vec{\Omega} \times [\vec{\Omega} \times (R_e\vec{e}_z + \vec{r})],$$

where R_e is the radius of the earth, $\vec{\Omega} = \Omega(\cos\lambda\, \vec{e}_y + \sin\lambda\, \vec{e}_z)$, $\Omega = 2\pi/(1 \text{ day})$, and λ is the latitude of Columbus (40° north).

(A) Draw a circle that represents a slice through the center of the earth that also passes through the north pole and through Columbus. Label these three points. Draw the vectors $\vec{\Omega}$, $R_e\vec{e}_z$, and $\vec{e}_y$ and indicate the angle λ. Given that $\vec{e}_x = \vec{e}_y \times \vec{e}_z$ points east, specify the directions of $\vec{e}_y$ and $\vec{e}_z$ using one of the words "up", "down", "north", "south", or "west".



$\vec{e}_Y$ points north

$\vec{e}_z$ points up

(B) A particle of mass m is subject only to the gravitational force $-mg\vec{e}_z$. Write down Newton's equations in the rotating frame. Identify the centrifugal force and the Coriolis force.

Newton's equation: $m\vec{a} = -mg\vec{e}_z$

$$m\left(\ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times[\vec{\Omega}\times(R_e\vec{e}_z + \vec{r})]\right) = -mg e_z$$

$$m\ddot{\vec{r}} = -mg\vec{e}_z \underbrace{-2m\,\vec{\Omega}\times\dot{\vec{r}}}_{\text{Coriolis force}} \underbrace{-\,m\,\vec{\Omega}\times[\vec{\Omega}\times(R_e\vec{e}_z+\vec{r})]}_{\text{centrifugal force}}$$

(C) Suppose a particle near the surface of the earth (so that $\vec{r}$ is negligible compared to $R_e\vec{e}_z$) is dropped from rest. What is its initial acceleration vector $\ddot{\vec{r}}(t = 0)$?

rest $\Rightarrow$ $\dot{\vec{r}}(0) = 0$

$$\ddot{\vec{r}}(0) = -g\vec{e}_z - \vec{\Omega}\times[\vec{\Omega}\times(R_e\vec{e}_z + \vec{r}(0))]$$

$$\approx -g\vec{e}_z - R_e\vec{\Omega}\times(\vec{\Omega}\times\vec{e}_z)$$

If we consider only gravity and the Coriolis force, the components of Newtons's equations reduce to

$$\begin{aligned}
\ddot{x} &= 2(\Omega \sin\lambda)\dot{y} - 2(\Omega\cos\lambda)\dot{z}, \\
\ddot{y} &= -2(\Omega\sin\lambda)\dot{x}, \\
\ddot{z} &= -g + 2(\Omega\cos\lambda)\dot{x}.
\end{aligned}$$

(D) If the particle is dropped from rest at a height h, the solution in the absence of the Coriolis force is

$$x_0(t) = 0, \qquad y_0(t) = 0, \qquad z_0(t) = h - \tfrac{1}{2}gt^2.$$

Simplify the 3 equations of motion, dropping terms that do not contribute at first order in Ω.

$$\dot{z}_0(t) = -gt$$

$$\ddot{x} = -2(\Omega\cos\lambda)\,\dot{z}_0(t) = 2g(\Omega\cos\lambda)\,t$$

$$\ddot{y} = 0$$

$$\ddot{z} = -g$$

(E) When it hits the ground ($z = 0$), the particle will have been deflected by the Coriolis force. Determine the direction and magnitude of its deflection to first order in Ω.

The solution to the equation for y is $y(t) = 0$, so there is no deflection in the y direction.

The solution to the equation for x can be obtained by integration: $\dot{x}(t) = 2g(\Omega\cos\lambda)\,t^2/2$

$$x(t) = 2g(\Omega\cos\lambda)\,t^3/6$$

The particle hits the ground at time T that satisfies $h - \frac{1}{2}gT^2 = 0$ or $T = \sqrt{2h/g}$

The deflection is $x(T) = g(\Omega\cos\lambda)T^3/3 = g(\Omega\cos\lambda)\,(2h/g)^{3/2}/3$

$$= \tfrac{2}{3}h(\Omega\cos\lambda)(2h/g)^{1/2}$$

(F) If the particle is thrown horizontally north from a height h with velocity v_0, the solution in the absence of the Coriolis force is

$$x_0(t) = 0, \qquad y_0(t) = v_0 t, \qquad z_0(t) = h - \tfrac{1}{2}gt^2.$$

Simplify the 3 equations of motion, dropping terms that do not contribute at first order in Ω.

$$\ddot{x} = 2(\Omega\sin\lambda)\dot{y}_0 - 2(\Omega\cos\lambda)\dot{z}_0$$

$$= 2\Omega[\sin\lambda\, v_0 + \cos\lambda\, gt]$$

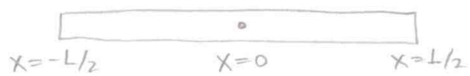
$$\ddot{y} = 0$$

$$\ddot{z} = -g$$

Problem 3.

A thin uniform rod of mass M and length L is suspended from one end of a string of length ℓ that is attached to a fixed support. Consider the motion of the rod in the x–z plane while the string remains taut. Take the point of support to be the origin: $(x, z) = (0, 0)$.

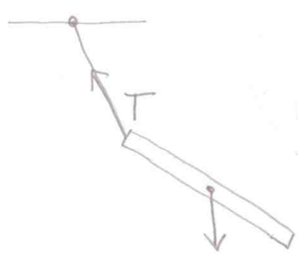
(A) Calculate the moment of inertia I of the rod about a perpendicular axis through its center by evaluating an integral.



$$I = \int_{-L/2}^{+L/2} x^2 \frac{M}{L} dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx = \frac{M}{L}\frac{x^3}{3}\Big|_{-L/2}^{+L/2} = \frac{M}{L}\frac{1}{3}\left(\frac{L}{2}\right)^3 \times 2$$

$$= \frac{1}{12} ML^2$$

(B) Sketch a generic configuration of the string and the rod and draw all the forces acting on the rod, including gravity. Which forces have unknown magnitudes? Which forces produce torques around the center of mass?



forces:

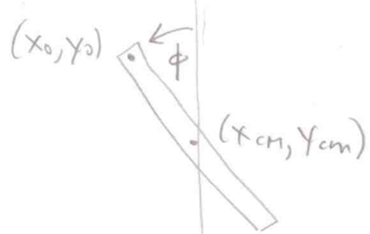
gravity: mg down

tension: T along the string

T has unknown magnitude

T produces a torque around the center of mass

(C) The configuration of the rod can be described by its center-of-mass coordinates $(x_{\rm cm}, z_{\rm cm})$ and by its rotation angle ϕ from the vertical. Determine the coordinates (x_0, z_0) of the end of the rod that is attached to the string.



$$x_0 = x_{cm} - \tfrac{1}{2}L\sin\phi$$
$$z_0 = z_{cm} + \tfrac{1}{2}L\cos\phi$$

(D) Express the condition that the string remains taut as a holonomic constraint on $x_{\rm cm}$, $z_{\rm cm}$, and ϕ.

$$x_0^2 + y_0^2 = \ell^2$$

$$\left(x_{cm} - \tfrac{1}{2}L\sin\phi\right)^2 + \left(z_{cm} + \tfrac{1}{2}L\cos\phi\right)^2 = \ell^2$$

(E) Express the kinetic energy K of the rod and its potential energy U in terms of x_{cm}, z_{cm}, ϕ, and their time derivatives.

$$K = \tfrac{1}{2}M(\dot{x}_{cm}^2 + \dot{z}_{cm}^2) + \tfrac{1}{2}I\dot{\phi}^2$$

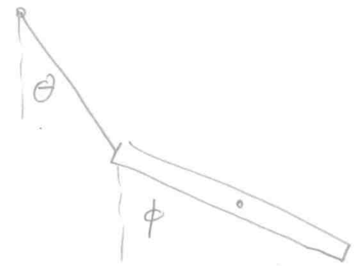
$$U = Mg\,z_{cm}$$

Let θ be the angle between the string and the vertical. (Recall that ϕ is the angle between the rod and the vertical.) We can use θ and ϕ as generalized coordinates for the system.

(F) Express x_{cm} and z_{cm} in terms of θ and ϕ.

$$x_{cm} = l\sin\theta + \tfrac{1}{2}L\sin\phi$$

$$z_{m} = -l\cos\theta - \tfrac{1}{2}L\cos\phi$$



(G) Express K and U in terms of θ, ϕ, and their time derivatives.

$$\dot{x}_{cm} = l\cos\theta\,\dot{\theta} + \tfrac{1}{2}L\cos\phi\,\dot{\phi} \qquad U = Mg(-l\cos\theta - \tfrac{1}{2}L\cos\phi)$$

$$\dot{z}_{cm} = l\sin\theta\,\ddot{\theta} + \tfrac{1}{2}L\sin\phi\,\dot{\phi}$$

$$K = \tfrac{1}{2}M(l\cos\theta\,\dot{\theta} + \tfrac{1}{2}L\cos\phi\,\dot{\phi})^2 + \tfrac{1}{2}M(l\sin\theta\,\dot{\theta} + \tfrac{1}{2}L\sin\phi\,\dot{\phi}) + \tfrac{1}{2}I\dot{\phi}^2$$

$$= \tfrac{1}{2}Ml^2\dot{\theta}^2 + \tfrac{1}{8}ML^2\dot{\phi}^2 + \tfrac{1}{2}MLl(\cos\theta\cos\phi + \sin\theta\sin\phi)\dot{\theta}\dot{\phi} + \tfrac{1}{2}I\dot{\phi}^2$$

(H) Write down the Lagrangian L for this system. Write down Lagrange's equations for θ and ϕ in terms of partial derivatives of L.

$$L = K - U$$

$$= \tfrac{1}{2}Ml^2\dot{\theta}^2 + \tfrac{1}{8}ML^2\dot{\phi}^2 + \tfrac{1}{2}MLl\cos(\theta-\phi)\dot{\theta}\dot{\phi} + \tfrac{1}{2}I\dot{\phi}^2 + Mg(l\cos\theta + \tfrac{1}{2}L\cos\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{\partial L}{\partial\phi}$$

(I) Write down Lagrange's equation for θ explicitly in terms of θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$. (K can be reduced to $\frac{1}{2}M[\ell^2\dot{\theta}^2 + \ell L\cos(\theta-\phi)\dot{\theta}\dot{\phi}]$ plus terms that do not depend on θ or $\dot{\theta}$.)

$$\frac{\partial L}{\partial\dot{\theta}} = \tfrac{1}{2}M[2l^2\dot{\theta} + lL\cos(\theta-\phi)\dot{\phi}]$$

$$\frac{\partial L}{\partial\theta} = -\tfrac{1}{2}MlL\sin(\theta-\phi)\dot{\theta}\dot{\phi} - Mgl\sin\theta$$

$$\frac{d}{dt}\left[Ml^2\dot{\theta} + \tfrac{1}{2}MLl\cos(\theta-\phi)\dot{\phi}\right] = -\tfrac{1}{2}MLl\sin(\theta-\phi)\dot{\theta}\dot{\phi} - Mgl\sin\theta$$