Physics 664 1st Midterm Exam NAME ______________________________

Problem 1.

Two particles with masses m_1 and m_2 and electric charges q_1 and q_2 interact through the electric force only. The force exerted on particle 1 by particle 2 is

$$\vec{F}_{1,2} = \frac{kq_1q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2).$$

(A) The potential energy of the pair of particles is

$$U = \frac{kq_1q_2}{|\vec{r}_1 - \vec{r}_2|}.$$

Express the time derivative $\dot{U}$ as a function of the positions $\vec{r}_1$ and $\vec{r}_2$ and the velocities $\dot{\vec{r}}_1$ and $\dot{\vec{r}}_2$.

$$U = \frac{kq_1q_2}{[(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_1-\vec{r}_2)]^{1/2}}$$

$$\dot{U} = -\frac{1}{2}\frac{kq_1q_2}{[(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_1-\vec{r}_2)]^{3/2}}\,2(\vec{r}_1-\vec{r}_2)\cdot(\dot{\vec{r}}_1-\dot{\vec{r}}_2)$$

$$= -\frac{kq_1q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)\cdot(\dot{\vec{r}}_1-\dot{\vec{r}}_2)$$

(B) The kinetic energy of the pair of particles is

$$K = \tfrac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 + \tfrac{1}{2}m_2\left|\dot{\vec{r}}_2\right|^2.$$

Express the time derivative $\dot{K}$ as a function of the position and velocity vectors.

$$K = \tfrac{1}{2}m_1(\dot{\vec{r}}_1\cdot\dot{\vec{r}}_1) + \tfrac{1}{2}m_2(\dot{\vec{r}}_2\cdot\dot{\vec{r}}_2)$$

$$\dot{K} = \tfrac{1}{2}m_1\,2(\dot{\vec{r}}_1\cdot\ddot{\vec{r}}_1) + \tfrac{1}{2}m_2\,2(\dot{\vec{r}}_2\cdot\ddot{\vec{r}}_2)$$

$$= \dot{\vec{r}}_1\cdot(m_1\ddot{\vec{r}}_1) + \dot{\vec{r}}_2\cdot(m_2\ddot{\vec{r}}_2) = \dot{\vec{r}}_1\cdot\left(\frac{kq_1q_2}{|\vec{r}_1-\vec{r}_2|}(\vec{r}_1-\vec{r}_2)\right) + \dot{\vec{r}}_2\cdot\left(\frac{kq_1q_2}{|\vec{r}_1-\vec{r}_2|}(\vec{r}_2-\vec{r}_1)\right)$$

$$= \frac{kq_1q_2}{|\vec{r}_1-\vec{r}_2|}(\dot{\vec{r}}_1-\dot{\vec{r}}_2)\cdot(\vec{r}_1-\vec{r}_2)$$

(C) What mathematical properties of the forces $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$ guarantee that the energy $K+U$ is conserved?

$\vec{F}_{1,2}$ and $\vec{F}_{2,1}$ are gradients of $-U$ with repect to $\vec{r}_1$ and $\vec{r}_2$

$$\vec{F}_{1,2} = -\nabla_1 U, \quad \vec{F}_{2,1} = -\nabla_2 U$$

(D) Newton's third law says that "for every action, there is an equal and opposite reaction". Express this law as an equation involving the forces $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$.

$$\vec{F}_{2,1} = -\vec{F}_{1,2}$$

(E) Express the total momentum $\vec{P}$ in terms of the position and velocity vectors. Derive the conservation of total momentum.

$$\vec{P} = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2$$

$$\dot{\vec{P}} = m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2$$

$$= \vec{F}_{1,2} + \vec{F}_{2,1}$$

$$= 0$$

(F) The total angular momentum is

$$\vec{L} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2.$$

Express the time derivative $\dot{\vec{L}}$ in terms of the forces $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$. What mathematical properties of the forces guarantee that $\vec{L}$ is conserved?

$$\dot{\vec{L}} = m_1 (\cancel{\dot{\vec{r}}_1 \times \dot{\vec{r}}_1} + \vec{r}_1 \times \ddot{\vec{r}}_1) + m_2 (\cancel{\dot{\vec{r}}_2 \times \dot{\vec{r}}_2} + \vec{r}_2 \times \ddot{\vec{r}}_2)$$

$$= \vec{r}_1 \times (m_1 \ddot{\vec{r}}_1) + \vec{r}_2 \times (m_2 \ddot{\vec{r}}_2)$$

$$= \vec{r}_1 \times \vec{F}_{1,2} + \vec{r}_2 \times \vec{F}_{2,1}$$

$$= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times (-\vec{F}_{1,2})$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{1,2}$$

$\vec{L}$ is conserved because $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$ are central forces i.e. they are both parallel to $\vec{r}_1 - \vec{r}_2$

Problem 2.
The coordinate $x(t)$ of a forced weakly-damped oscillator satisfies the real differential equation

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = f_0 \cos(\omega t),$$

where $\nu \ll \omega_0$.

(A) Suppose the complex-valued function $z(t)$ satisfies the complex differential equation

$$\ddot{z} + 2\nu\dot{z} + \omega_0^2 z = f_0 \exp(i\omega t).$$

How can $z(t)$ be used to construct a real-valued solution $x(t)$ to the real differential equation?

$x(t) = \mathrm{Re}\, z(t)$

(B) Find a particular solution $z_p(t)$ to the complex differential equation.

look for a solution of the form $z(t) = A e^{i\omega t}$

$\dot{z} = A(i\omega)e^{i\omega t}$

$\ddot{z} = A(-\omega^2)e^{i\omega t}$

$[(-\omega^2) + 2\nu(i\omega) + \omega_0^2] A e^{i\omega t} = f_0 e^{i\omega t}$

$A = \dfrac{f_0}{-\omega^2 + 2i\nu\omega + \omega_0^2}$

$$\boxed{z_p(t) = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\nu\omega} e^{i\omega t}}$$

(C) Find a particular solution $x_p(t)$ to the real differential equation.

$$x_p(t) = \mathrm{Re}\, z_p(t)$$
$$= \mathrm{Re}\left(\frac{f_0}{\omega_0^2 - \omega^2 + 2i\nu\omega} e^{i\omega t}\right)$$
$$= \mathrm{Re}\left(\frac{f_0}{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}(\omega_0^2 - \omega^2 - 2i\nu\omega)\left[\cos(\omega t) + i\sin(\omega t)\right]\right)$$

$$\boxed{x_p(t) = \frac{f_0}{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}\left[(\omega_0^2-\omega^2)\cos(\omega t) + 2\nu\omega\sin(\omega t)\right]}$$

(D) The equation for $x(t)$ in the absence of the force is

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = 0.$$

Find the most general real solution to this equation.

look for solutions of the form $x(t) = A e^{i\alpha t}$

$\dot{x} = A(i\alpha)e^{i\alpha t}$

$\ddot{x} = A(-\alpha^2)e^{i\alpha t}$

$[(-\alpha^2) + 2\nu(i\alpha) + \omega_0^2] A e^{i\alpha t} = 0$

$-\alpha^2 + 2i\nu\alpha + \omega_0^2 = 0 \implies \alpha = \dfrac{-2i\nu \pm \sqrt{-4\nu^2 + 4\omega_0^2}}{2(-1)} = i\nu \pm \sqrt{\omega_0^2 - \nu^2}$

independent solutions are $e^{i(i\nu \pm \sqrt{\omega_0^2-\nu^2})t} = e^{-\nu t} e^{\pm i\sqrt{\omega_0^2-\nu^2}\,t}$

most general real solution: $\boxed{x(t) = A e^{-\nu t}\cos\left(\sqrt{\omega_0^2-\nu^2}\,t\right) + B e^{-\nu t}\sin\left(\sqrt{\omega_0^2-\nu^2}\,t\right)}$

(E) Given the particular solution $x_p(t)$ from part (C), write down the most general solution $x(t)$ to the real differential equation for the forced damped oscillator. How does this solution simplify at very large times t?

$$X(t) = X_p(t) + A e^{-\nu t} \cos\left(\sqrt{\omega_0^2-\nu^2}\, t\right) + B e^{-\nu t} \sin\left(\sqrt{\omega_0^2-\nu^2}\, t\right)$$

as $t \to \infty$, $e^{-\nu t} \to 0$

$$X(t) \to X_p(t)$$

(F) After the transients die away, the solution $x(t)$ oscillates with the angular frequency ω of the driving force. What is the amplitude of the oscillation? For approximately what driving frequency is the amplitude largest? Write down (but don't solve) an equation that determines the exact frequency ω at which the amplitude is largest.

$$Z_p(t) = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\nu\omega} e^{i\omega t}$$

$$= \left| \frac{f_0}{\omega_0^2 - \omega^2 + 2i\nu\omega} \right| e^{i\phi} e^{i\omega t} \quad \text{where } \phi = \arctan \frac{2\nu\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}} e^{i(\omega t + \phi)}$$

$$X_p(t) = \operatorname{Re} Z_p(t)$$

$$= \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}} \cos(\omega t + \phi)$$

amplitude: $\dfrac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}}$

since $\nu \ll \omega_0$, maximum is near $\omega = \omega_0$

exact frequency of maximum amplitude satisfies $\dfrac{d}{d\omega} \dfrac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\nu\omega)^2}} = 0$

Problem 3.

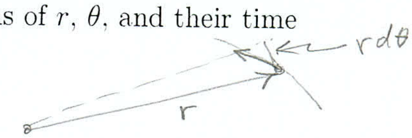
Two bodies with masses M and m interact through gravitational forces. Newton's equations for the separation vector $\vec{r}(t)$ can be reduced to equations of motion for the polar coordinates $r(t)$ and $\theta(t)$:

$$\begin{aligned}\mu\left(\ddot{r}+r\dot{\theta}^2\right) &= -\frac{GMm}{r^2}.\\ \mu\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) &= 0.\end{aligned}$$

The conserved angular momentum L and the conserved energy E are

$$\begin{aligned}L &= \mu r^2\dot{\theta}.\\ E &= \tfrac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{GMm}{r}.\end{aligned}$$

(A) Kepler's 2nd Law states that $\vec{r}(t)$ sweeps out equal areas in equal times. Express the time derivative $\dot{A}$ of the area $A(t)$ swept out between times 0 and t in terms of r, θ, and their time derivatives. Express $\dot{A}$ in terms of conserved quantities.



differential area : $dA = \frac{1}{2} r (r\,d\theta)$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2}\frac{L}{\mu}$$

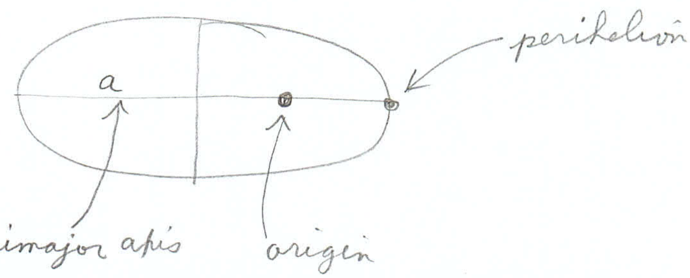
(B) The most general equation for the orbit (up to a rotation) is

$$r(\theta)=\frac{r_0}{1+e\cos\theta},$$

where $r_0 = L^2/GMm\mu$ and $0 \le e < \infty$. The orbits for $0<e<1$ are ellipses, in agreement with Kepler's 1st Law. Determine the radius r_p at perihelion and the semimajor axis a. Sketch the orbit for $e = 1/2$ (which implies $r_{\max}/r_{\min} = 3$). Label the origin, the perihelion, and the semimajor axis a.

$$r_p = r_{min} = \frac{r_0}{1+e}$$

$$a = \frac{1}{2}\left(r_{min}+r_{max}\right) = \frac{1}{2}\left(\frac{r_0}{1+e}+\frac{r_0}{1-e}\right) = \frac{r_0}{1-e^2}$$



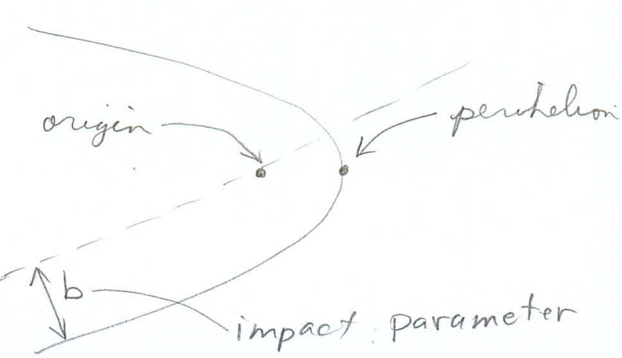
(C) The orbits for $1<e<\infty$ are hyperbolas. The angle $\theta(t)$ approaches finite limits $\theta_\pm$ as $t\to\pm\infty$. Determine the radius r_p at perihelion and the angles θ_- and θ_+. Sketch a hyperbolic orbit, labelling the origin, the perihelion, and the impact parameter b.

$$r_p = r_{min} = \frac{r_0}{1+e}$$

$$r = \infty \Rightarrow 1+e\cos\theta_\pm = 0$$

$$\cos\theta_\pm = -\frac{1}{e}$$

$$\theta_\pm = \pm\arccos\left(-\frac{1}{e}\right)$$



(D) Use the conservation laws to express $\dot{r}$ and $\dot{\theta}$ as functions of r only. Then use these equations to derive a first order differential equation for the orbit $r(\theta)$.

$$L = \mu r^2 \dot{\theta} \longrightarrow \dot{\theta} = \frac{L}{\mu r^2}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GMm}{r} \Longrightarrow \dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} + \frac{GMm}{r}\right)}$$

$$\left.\begin{aligned} \dot{\theta} &= \frac{d\theta}{dt} \\ \dot{r} &= \frac{dr}{dt} \end{aligned}\right\} \quad \frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta} \qquad \boxed{\frac{dr}{d\theta} = \pm \frac{\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} + \frac{GMm}{r}\right)}}{L/\mu r^2}}$$

(E) Given the orbit $r(\theta)$ in part (B), angular momentum conservation can be used to derive a first order differential equation for $\theta(t)$:

$$\dot{\theta} = \frac{L}{\mu r_0^2}(1 + e\cos\theta)^2.$$

Given the initial condition $\theta(0) = 0$, solve the differential equation by quadrature to obtain t as a function of θ. (Your solution should be left as a definite integral with θ as one of the endpoints.)

$$\dot{\theta} = \frac{d\theta}{dt} \qquad \frac{d\theta}{dt} = \frac{L}{\mu r_0^2}(1 + e\cos\theta)^2$$

$$dt = \frac{\mu r_0^2}{L}\frac{d\theta}{(1+e\cos\theta)^2}$$

$$\int_0^t dt' = \frac{\mu r_0^2}{L}\int_0^\theta d\theta' \frac{1}{(1+e\cos\theta')^2}$$

$$\boxed{t = \frac{\mu r_0^2}{L}\int_0^\theta d\theta' \frac{1}{(1+e\cos\theta')^2}}$$

(F) The period T of an elliptical orbit is

$$T = \frac{2\pi a^{3/2}}{\sqrt{G(M+m)}}$$

Kepler's 3rd Law states that T^2/a^3 is the same for all planets. Explain in what sense this Law is only approximate for the solar system (in contrast to Kepler's 1st and 2nd Laws, which are exact).

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}$$

This is not the same for all planets, because it depends on the mass m of the planets.

In the solar system, the masses of all the planets are much smaller than the mass M of the sun.

So T^2/a^3 is almost the same for all planets.