Physics 664 2nd Midterm Exam NAME ____________________________

1. A particle of mass m slides along the smooth upper surface of a cylinder of radius R in a constant gravitational field. Choose Cartesian coordinates (x, y) so that the surface of the cylinder is $x^2 + y^2 = R^2$ and the acceleration of the gravitational field is $-g\hat{y}$. For simplicity, consider only motion of the particle in the x–y plane.

(a) Express the condition that the normal force keeps the particle on the surface of the cylinder as a holonomic constraint on the Cartesian coordinates (x, y). Express it as a holonomic constraint on the polar coordinates (ρ, θ) defined by $x = \rho\cos\theta$ and $y = \rho\sin\theta$.

$$x^2+y^2-R^2=0$$

$$\rho-R=0$$

(b) Write down the kinetic energy T and the potential energy V of the particle using Cartesian coordinates. Write them down using polar coordinates.

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\theta}^2)$$

$$V=mgy = mg\rho\sin\theta$$

(c) As long as the particle remains on the surface of the cylinder, its position can be specified by the polar angle θ. Write down the Lagrangian for the particle in terms of the coordinate θ and its time derivative $\dot{\theta}$.

$$L=\frac{1}{2}mR^2\dot{\theta}^2 - mg\rho\sin\theta$$

(d) If it is moving on the surface of the cylinder, the particle must be accelerating towards the axis of the cylinder. Express its acceleration in terms of θ and $\dot{\theta}$.

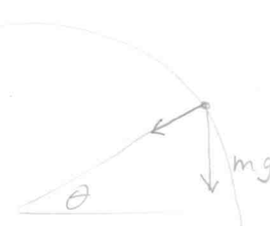
$$\frac{v^2}{\rho} = \frac{(\rho\dot{\theta})^2}{\rho} = \rho\dot{\theta}^2 = R\dot{\theta}^2$$

(e) The particle can fly off the surface of the cylinder if its velocity becomes too large. Use Newton's equation for the radial coordinate ρ to deduce the conditions on θ and $\dot{\theta}$ for the effects of the normal force to be treated as a holonomic constraint.

$$mR\dot{\theta}^2 = mg\sin\theta - N$$

$$N = mg\sin\theta - mR\dot{\theta}^2$$

$$N>0 \implies g\sin\theta \geq R\dot{\theta}^2$$



2. The Kepler problem for a planet with orbit in the x–y plane can be summarized by the following Lagrangian for its polar coordinates (r, θ):

$$L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r},$$

where $\mu = Mm/(M+m)$ is the reduced mass.

(a) Write down the expression for the energy E of the planet in terms of the polar coordinates r and θ. What property of the Lagrangian guarantees that E is conserved?

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$$

L does not depend explicitly on t

(b) Write down the expression for the z–component of the angular momentum L_z of the planet in terms of the polar coordinates r and θ. What property of the Lagrangian guarantees that L_z is conserved?

$$L_z = \mu r^2 \dot{\theta}$$

L is invariant under $\theta \to \theta + \alpha$

(c) Use conservation laws to reduce the equations of motion to two first-order differential equations for r and θ of the form

$$\dot{r} = f(r, \theta), \qquad \dot{\theta} = g(r, \theta).$$

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E + \frac{GMm}{r} - \frac{L_z^2}{2\mu r^2}\right)}$$

$$\dot{\theta} = \frac{L_z}{\mu r^2}$$

(d) Use the first-order differential equations for $r(t)$ and $\theta(t)$ to derive a differential equation for the orbit $r(\theta)$.

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{f(r,\theta)}{g(r,\theta)}$$

(e) Use conservation laws to derive Kepler's 2nd law: the rate at which area is swept out by the position vector of the planet is constant in time.

$$dA = \frac{1}{2} r^2 d\theta$$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2\mu}(\mu r^2 \dot{\theta}) = \frac{L_z}{2\mu} \quad \text{constant}$$

3. The Lagrangian for a charged particle in a constant magnetic field $B_0\hat{z}$ can be written

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \tfrac{1}{2}B_0\,(\dot{x}y - \dot{y}x)\,.$$

(a) Write down Lagrange's equation for the coordinate x in as explicit a form as possible.

$$\frac{\partial L}{\partial x} = \frac{1}{2}B_0\dot{y} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{1}{2}B_0 y \qquad m\ddot{x} - \frac{1}{2}B_0\dot{y} = \frac{1}{2}B_0\dot{y}$$

$$m\ddot{x} - B_0\dot{y} = 0$$

(b) Express the Hamiltonian H as a function of the coordinates (x, y, z) and their time derivatives.

$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

(c) Determine the canonical momenta (p_x, p_y, p_z) that are conjugate to (x, y, z).

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{1}{2}B_0 y$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{1}{2}B_0 x$$

$$p_z = \frac{\partial L}{\partial \dot{z}}$$

(d) Express the Hamiltonian H as a function of the canonical momenta (p_x, p_y, p_z) and the coordinates (x, y, z).

$$H = \frac{1}{2}m\left[\left(\frac{p_x + \frac{1}{2}B_0 y}{m}\right)^2 + \left(\frac{p_y - \frac{1}{2}B_0 x}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right]$$

$$= \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2 + B_0(yp_x - xp_y) + \frac{1}{4}B_0^2(x^2+y^2)\right]$$

(e) Write down Hamilton's equations for the coordinate x and for the canonical momentum p_x in as explicit a form as possible.

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{1}{2}B_0 y\right)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{1}{m}\left(p_y - \frac{1}{2}B_0 x\right)\left(-\frac{1}{2}B_0\right)$$