Spring 2009

Physics 664 Midterm Exam NAME ______________________

1. The driven damped pendulum is described by a differential equation of the form

$$\ddot{\theta} + 2\beta\dot{\theta} + \sin\theta = f\cos(\omega t).$$

Suppose the values of f and ω are fixed and the damping parameter β is varied. For very large β, the attractor for trajectories with initial conditions near $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ is a periodic trajectory with the same period $\tau = 2\pi/\omega$ as the driving force. As β decreases, the attractor changes to period 2τ at $\beta_1 = 1.4$ and then to period 4τ at $\beta_2 = 1.0$. As β decreases further, there are further period doublings followed by the onset of chaos when β decreases below some value β_∞. (Feigenbaum's constant is $F \approx 4.669$, but for the purposes of this problem, please approximate it by $F \approx 5$.)

(a) Estimate the value β_3 at which the period changes from 4τ to 8τ. (Use $F \approx 5$.)

(b) Estimate the value β_∞ at which chaos sets in. (Use $F \approx 5$.)

(c) Draw plausible phase space trajectories for the attractors at $\beta = 1.01$ and $\beta = 0.99$. (Be sure to label the axes of the two plots.)

(d) Draw plausible Poincare sections for the attractors at $\beta = 1.01$ and $\beta = 0.99$. (Be sure to label the axes of the two plots.)

See Fitzpatrick, Chapter 15

2. The fundamental symmetries of Newtonian mechanics include translations, rotations, and Galilean boosts. The Cartesian coordinates for the path of a particle are $(x(t), y(t), z(t))$. Specify the Cartesian coordinates $(x'(t), y'(t), z'(t))$ for the new path obtained from each of the following symmetries:

(a) translation by a distance a along the x axis,

$x'(t) = x(t) + a$
$y'(t) = y(t)$
$z'(t) = z(t)$

(b) rotation by infinitesimal angle θ around the y axis,

$x'(t) = x(t) - \theta z(t)$
$y'(t) = y(t)$
$z'(t) = z(t) + \theta x(t)$

(c) Galilean boost with velocity w in the direction of the z axis,

$x'(t) = x(t)$
$y'(t) = y(t)$
$z'(t) = z(t) + wt$

Newton's equations for the motion of a particle in the earth's gravitational field are

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{z}} \qquad \text{for } z > 0,$$

where we have chosen the surface of the earth to be at $z = 0$. For each of the following fundamental symmetries, there are 3 independent transformations. Identify those that are symmetries of this system.

(d) translations:

translate along x axis
" y axis

(e) rotations:

rotate around z axis

(f) Galilean boosts:

boost along x axis
" y axis

3. The linear driven damped harmonic oscillator is described by the differential equation

$$m\ddot{x} + 2B\dot{x} + kx = F\cos(\omega t).$$

(a) The sum of the kinetic and potential energies of the oscillator is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

Derive an expression for the instantaneous rate of change of energy dE/dt that shows that it comes only from the damping and driving terms.

$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x}$$

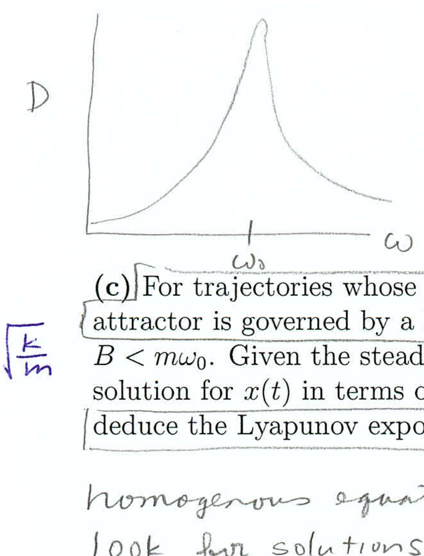
$$= \dot{x}\left[-2B\dot{x} - \cancel{kx} + F\cos(\omega t)\right] + \cancel{kx\dot{x}}$$

$$= \underbrace{-2B\dot{x}^2}_{\text{from damping}} + \underbrace{F\dot{x}\cos(\omega t)}_{\text{from forcing}}$$

(b) The steady-state solution, which is an attractor for trajectories with all possible initial conditions, has the form

$$x(t) = D\cos(\omega t + \delta).$$

Sketch the amplitude D as a function of the driving frequency ω, labelling the natural frequency $\omega_0 = \sqrt{k/m}$. Derive (but don't solve) a system of two algebraic equations in two unknowns that determine D as a function of ω.



derivation using complex algebra is simpler!

$x(t) = \text{Re}\, z(t)$, where $z(t)$ satisfies

$m\ddot{z} + 2B\dot{z} + kz = Fe^{i\omega t}$

look for solution $z(t) = Ae^{i\omega t}$

$[m(-\omega^2) + 2B(i\omega) + k]A e^{i\omega t} = Fe^{i\omega t}$

$\Longrightarrow A = \dfrac{F}{-m\omega^2 + 2i\omega B + k} \Longrightarrow D = \dfrac{F}{|-m\omega^2 + 2i\omega B + k|}$

(c) For trajectories whose initial conditions are near the attractor, the approach to the attractor is governed by a Luyapunov exponent λ. SKIP! Consider the underdamped case $B < m\omega_0$. Given the steady-state solution in part (b), write down the most general solution for $x(t)$ in terms of two unknown real-valued constants. Use this solution to deduce the Lyapunov exponent for this system. SKIP!

$\omega_0 \equiv \sqrt{\frac{k}{m}}$

homogenous equation: $m\ddot{x} + 2B\dot{x} + k = 0$

look for solutions $x(t) = x_0 e^{i\omega t}$: $[m(-\omega^2) + 2B(i\omega) + k]\, x_0 e^{i\omega t} = 0$

$\omega^2 - \frac{2iB}{m}\omega - \frac{k}{m} = 0$ $\qquad \omega = \dfrac{\frac{2iB}{m} \pm \sqrt{-\frac{4B^2}{m^2} + 4k/m}}{2}$

$= i\frac{B}{m} \pm \sqrt{\omega_0^2 - B^2/m^2}$

$e^{i\omega t} = e^{-(B/m)t}\, e^{\pm i\sqrt{\omega_0^2 - B^2/m^2}\, t}$

general solution: $x(t) = C_1 e^{-(B/m)t}\cos\left(\sqrt{\omega_0^2 - B^2/m^2}\, t\right) + C_2 e^{-(B/m)t}\sin\left(\sqrt{\omega_0^2 - B^2/m^2}\, t\right)$

where C_1, C_2 are arbitrary real constants