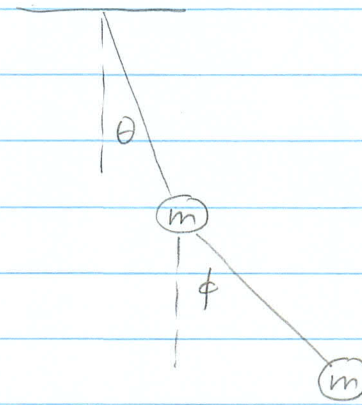
Fitzpatrick, Chapter 9

Exercise 9.2



The coordinate of the two masses are

$$x_1 = l \sin\theta$$

$$y_1 = -l\cos\theta$$

$$x_2 = l\sin\theta + l\sin\phi$$

$$y_2 = -l\cos\theta - l\cos\phi$$

Their time derivative are

$$\dot{x}_1 = l\cos\theta\,\dot{\theta}$$

$$\dot{y}_1 = l\sin\theta\,\dot{\theta}$$

$$\dot{x}_2 = l\cos\theta\,\dot{\theta} + l\cos\phi\,\dot{\phi}$$

$$\dot{y}_2 = l\sin\theta\,\dot{\theta} + l\sin\phi\,\dot{\phi}$$

The potential energy is

$$V = mgy_1 + mgy_2$$

$$= -mgl(2\cos\theta + \cos\phi)$$

The kinetic energy is

$$K = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \tfrac{1}{2}ml^2\Big[(\cos\theta\,\dot{\theta})^2 + (\sin\theta\,\dot{\theta})^2$$

$$+ (\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi})^2 + (\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi})^2\Big]$$

$$= \tfrac{1}{2}ml^2\left[2\dot{\theta}^2 + 2(\cos\theta\cos\phi + \sin\theta\sin\phi)\dot{\theta}\dot{\phi} + \dot{\phi}^2\right]$$

$$= \tfrac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\cos(\theta-\phi)\dot{\theta}\dot{\phi}\right]$$

Thus the Lagrangian is

$$L = K - V$$

$$= \tfrac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\cos(\theta-\phi)\dot{\theta}\dot{\phi}\right]$$

$$+ mgl\left[2\cos\theta + \cos\phi\right]$$

The derivative of L are

$$\frac{\partial L}{\partial \theta} = \tfrac{1}{2}ml^2\left[-2\sin(\theta-\phi)\dot{\theta}\dot{\phi}\right] + mgl\left[-2\sin\theta\right]$$

$$\frac{\partial L}{\partial \phi} = \tfrac{1}{2}ml^2\left[+2\sin(\theta-\phi)\dot{\theta}\dot{\phi}\right] + mgl\left[-\sin\theta\right]$$

$$\frac{\partial L}{\partial \dot{\theta}} = \tfrac{1}{2}ml^2\left[4\dot{\theta} + 2\cos(\theta-\phi)\dot{\phi}\right]$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} m l^2 \left[2\dot{\phi} + 2\cos(\theta - \phi)\,\dot{\theta} \right]$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(m l^2 \left[2\dot{\theta} + \cos(\theta - \phi)\,\dot{\phi} \right] \right)$$

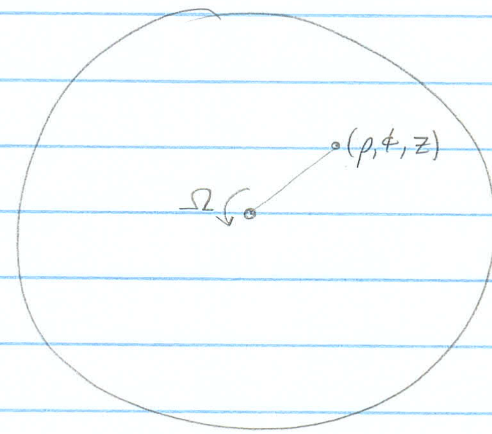
$$= -m l^2 \sin(\theta - \phi)\,\dot{\theta}\dot{\phi} - 2 m g l \sin\theta$$

$$\frac{d}{dt}\left(m l^2 \left[\dot{\phi} + \cos(\theta - \phi)\,\dot{\theta} \right] \right)$$

$$= m l^2 \sin(\theta - \phi)\,\dot{\theta}\dot{\phi} - m g l \sin\theta$$

Fitzpatrick, Chapter 9

Exercise 9.3



The Cartesian coordinates of the particle in the inertial frame expressed in terms of corotating cylindrical polar coordinates are

$$x = \rho \cos(\Omega t + \phi)$$

$$y = \rho \sin(\Omega t + \phi)$$

$$z = z$$

Their time derivatives are

$$\dot{x} = \dot{\rho} \cos(\Omega t + \phi) - \rho \sin(\Omega t + \phi)(\Omega + \dot{\phi})$$

$$\dot{y} = \dot{\rho} \sin(\Omega t + \phi) + \rho \cos(\Omega t + \phi)(\Omega + \dot{\phi})$$

$$\dot{z} = \dot{z}$$

The potential energy is

$$V = mgz$$

The kinetic energy is

$$K = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \tfrac{1}{2}m\Big(\dot{\rho}^2\left[\cos^2(\Omega t + \phi) + \sin^2(\Omega t+\phi)\right]$$

$$+ \rho^2(\Omega + \dot{\phi})^2\left[\sin^2(\Omega t+\phi) + \cos^2(\Omega t+\phi)\right]$$

$$+ 2\rho\dot{\rho}(\Omega+\dot{\phi})\cos(\Omega t+\phi)\sin(\Omega t+\phi)\left[-1+1\right]$$

$$+ \dot{z}^2\Big)$$

$$= \tfrac{1}{2}m\left[\dot{\rho}^2 + \rho^2(\Omega+\dot{\phi})^2 + \dot{z}^2\right]$$

The Lagrangian is therefore

$$L = K - V$$

$$= \tfrac{1}{2}m\left[\dot{\rho}^2 + \rho^2(\Omega+\dot{\phi})^2 + \dot{z}^2\right] - mgz$$

The conjugate momenta are

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2(\Omega + \dot{\phi})$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

The Lagrangian does not depend on ϕ.
Therefore p_ϕ is conserved

The derivative of L with respect to the coordinate are

$$\frac{\partial L}{\partial \rho} = \frac{1}{2}m\left[2\rho(\Omega+\dot{\phi})^2\right]$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial z} = -mg$$

Lagrange's equation of motion are

$$\frac{d}{dt}\left[m\dot{\rho}\right] = m\rho(\Omega+\dot{\phi})^2$$

$$\frac{d}{dt}\left[m\rho^2(\Omega+\dot{\phi})\right] = 0$$

$$\frac{d}{dt}\left[m\dot{z}\right] = -mg$$

They can be simplified to

$$\ddot{\rho} = \rho(\Omega+\dot{\phi}^2)$$

$$\rho^2\ddot{\phi} + 2\rho\dot{\rho}(\Omega+\dot{\phi}) = 0$$

$$\ddot{z} = -g$$

where λ is small, and $\omega_{y'}$ is initially small. Euler's equations (with no applied torques) take the form

$$\begin{aligned} I_\perp\,\dot{\omega}_{x'} - (I_\perp - I_\parallel)\,\omega_{y'}\,\lambda &= 0, && (31)\\ I_\perp\,\dot{\omega}_{y'} - (I_\parallel - I_\perp)\,\lambda\,\omega_{x'} &= 0, && (32)\\ I_\parallel\,\dot{\lambda} &= 0. && (33) \end{aligned}$$

It follows that $\dot{\lambda} = 0$. Moreover, (31) and (32) can be written

$$\begin{aligned} \dot{\omega}_{x'} &= k\,\omega_{y'}, && (34)\\ \dot{\omega}_{y'} &= -k\,\omega_{x'}, && (35) \end{aligned}$$

where

$$k = (1 - I_\parallel/I_\perp)\,\lambda. \tag{36}$$

The solution is

$$\begin{aligned} \omega_{x'} &= a\,\cos(k\,t), && (37)\\ \omega_{y'} &= a\,\sin(k\,t), && (38) \end{aligned}$$

where a and b are constants. We conclude that the direction of the $\boldsymbol{\omega}$ vector very slowly rotates in the x'-y' plane at a rate proportional to λ. In this respect, the body is not stable to rotation about the x'-axis, since even if $\omega_{y'}$ is initially small it does not remain small indefinitely. By symmetry, the body is also not stable to rotation about the y'-axis.

9.4

4. Let θ be the angle that the string subtends with the downward vertical, and let x be the extension of the string. The kinetic energy of the system is

$$K = \frac{1}{2}\,m\left[(l_0 + x)^2\,\dot{\theta}^{\,2} + \dot{x}^{\,2}\right], \tag{39}$$

whereas the potential energy takes the form

$$U = -m\,g\,(l_0 + x)\,\cos\theta + \frac{1}{2}\,k\,x^2. \tag{40}$$

Therefore, the Lagrangian is

$$L = \frac{1}{2}\,m\left[(l_0 + x)^2\,\dot{\theta}^{\,2} + \dot{x}^{\,2}\right] + m\,g\,(l_0 + x)\,\cos\theta - \frac{1}{2}\,k\,x^2. \tag{41}$$

Now, Lagrange's equations of motion for the system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0. \tag{43}$$

So,

$$\frac{d}{dt}[m\,\dot{x}] - m\,(l_0 + x)\,\dot{\theta}^{\,2} - m\,g\,\cos\theta + k\,x = 0, \tag{44}$$

$$\frac{d}{dt}\left[m\,(l_0 + x)^2\,\dot{\theta}\right] + m\,g\,(l_0 + x)\,\sin\theta = 0, \tag{45}$$

giving

$$\ddot{x} - (l_0 + x)\,\dot{\theta}^{\,2} = g\,\cos\theta - \frac{k}{m}\,x, \tag{46}$$

$$\ddot{\theta} + \frac{2\,\dot{x}\,\dot{\theta}}{l_0 + x} = -\frac{g}{l_0 + x}\,\sin\theta. \tag{47}$$

5. The mass' perpendicular distance from the axis of rotation is $l\,\sin\theta$, and its vertical distance below point O is $l\,\cos\theta$. The mass' kinetic energy is thus

$$K = \frac{1}{2}\,m\,(l^2\,\dot{\theta}^{\,2} + l^2\,\sin^2\theta\,\omega^2), \tag{48}$$

and its potential energy takes the form

$$U = -m\,g\,l\,\cos\theta. \tag{49}$$

Thus, the Lagrangian is

$$L = \frac{1}{2}\,m\,(l^2\,\dot{\theta}^{\,2} + l^2\,\sin^2\theta\,\omega^2) + m\,g\,l\,\cos\theta, \tag{50}$$

and Lagrange's equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{51}$$

yields

$$\frac{d}{dt}\left(m\, l^2\, \dot{\theta}\right) - m\, l^2\, \cos\theta\, \sin\theta\, \omega^2 + m\, g\, l\, \sin\theta = 0, \tag{52}$$

or

$$\frac{d^2\theta}{dt^2} - \omega^2\, \sin\theta\, \cos\theta = -\frac{g}{l}\, \sin\theta. \tag{53}$$

Suppose that θ is small. The above equation reduces to

$$\frac{d^2\theta}{dt^2} \simeq -\left(\frac{g}{l} - \omega^2\right)\theta. \tag{54}$$

Thus, the position $\theta = 0$ is stable (*i.e.*, the above equation has oscillatory, rather than exponential, solutions) provided that

$$\omega^2 < \frac{g}{l}. \tag{55}$$

9.6 6. The mass has coordinates

$$x = a\, \cos(\omega\, t) + a\, \cos(\omega\, t + \theta), \tag{56}$$

$$z = a\, \sin(\omega\, t) + a\, \sin(\omega\, t + \theta). \tag{57}$$

Hence,

$$\dot{x} = -a\, \omega\, \sin(\omega\, t) - a\, (\omega + \dot{\theta})\, \sin(\omega\, t + \theta), \tag{58}$$

$$\dot{z} = a\, \omega\, \cos(\omega\, t) + a\, (\omega + \dot{\theta})\, \cos(\omega\, t + \theta). \tag{59}$$

The kinetic energy of the mass is thus

$$K = \frac{1}{2}\, m\, (\dot{x}^{\,2} + \dot{z}^{\,2}) = \frac{1}{2}\, m\, a^2 \left[\omega^2 + 2\, \omega\, (\omega + \dot{\theta})\, \cos\theta + (\omega + \dot{\theta})^{\,2}\right]. \tag{60}$$

The potential energy of the mass is

$$U = m\, g\, z = m\, g\, a \left[\sin(\omega\, t) + \sin(\omega\, t + \theta)\right]. \tag{61}$$

Thus, the Lagrangian takes the form

$$\begin{aligned} L &= \frac{1}{2}\, m\, a^2 \left[\omega^2 + 2\,\omega\,(\omega + \dot{\theta})\,\cos\theta + (\omega + \dot{\theta})^{\,2}\right] \\ &\quad - m\,g\,a\,[\sin(\omega\,t) + \sin(\omega\,t + \theta)]\,. \end{aligned} \tag{62}$$

Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{63}$$

which gives

$$\begin{aligned} 0 &= \frac{d}{dt}\left[m\,a^2\,\omega\,\cos\theta + m\,a^2\,(\omega + \dot{\theta})\right] \\ &\quad + \left[m\,a^2\,\omega\,(\omega + \dot{\theta})\,\sin\theta - m\,g\,a\,\cos(\omega\,t + \theta)\right], \end{aligned} \tag{64}$$

or

$$\ddot{\theta} + \omega^2\,\sin\theta + \frac{g}{a}\,\cos(\omega\,t + \theta) = 0. \tag{65}$$

In the limit that $1 \gg |\theta| \gg g/(a\,\omega^2)$ the above equation reduces to

$$\ddot{\theta} + \omega^2\,\theta \simeq 0. \tag{66}$$

Thus, the mass executes simple harmonic oscillation about the point $\theta = 0$ at the angular frequency ω. The mass acts like a simple pendulum of effective length g/ω^2.

Physics 336K: Newtonian Dynamics
Homework 8: Solutions

9.8 1. If the object is freely rotating then $U = 0$ and $L = K$. It follows that

$$L = \frac{1}{2}\left[I_\perp\,\dot{\theta}^{\,2} + (I_\perp\,\sin^2\theta + I_\parallel\,\cos^2\theta)\,\dot{\phi}^{\,2} + 2\,I_\parallel\,\cos\theta\,\dot{\phi}\,\dot{\psi} + I_\parallel\,\dot{\psi}^{\,2}\right]. \qquad (1)$$

The fact that L is not an explicit function of ψ implies that

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = I_\parallel\,(\cos\theta\,\dot{\phi} + \dot{\psi}) \qquad (2)$$

is a constant of the motion. The fact that L is not an explicit function of ϕ implies that

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = (I_\perp\,\sin^2\theta + I_\parallel\,\cos^2\theta)\,\dot{\phi} + I_\parallel\,\cos\theta\,\dot{\psi} \qquad (3)$$

is a constant of the motion. The Lagrangian equations of motion are

$$p_\psi = \text{constant}, \qquad (4)$$

$$p_\phi = \text{constant}, \qquad (5)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta}. \qquad (6)$$

The latter equation reduces to

$$I_\perp\,\ddot{\theta} + \left[I_\parallel\,\dot{\psi} + (I_\parallel - I_\perp)\,\cos\theta\,\dot{\phi}\right]\sin\theta\,\dot{\phi} = 0. \qquad (7)$$

For steady precession, in which θ, $\dot{\phi}$, and $\dot{\psi}$ are constants, the above equation yields

$$\dot{\psi} = \left(\frac{I_\perp - I_\parallel}{I_\parallel}\right)\cos\theta\,\dot{\phi}. \qquad (8)$$