and M is the mass of the Earth. The equation of motion of the mass in a frame co-moving with the satellite is

$$\ddot{\mathbf{r}} = -G\,M\,\frac{(\mathbf{r}-\mathbf{a})}{(r^2 - 2\,\mathbf{r}\cdot\mathbf{a} + a^2)^{3/2}} - \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}-\mathbf{a})] - 2\,\boldsymbol{\omega}\times\dot{\mathbf{r}}, \tag{46}$$

where $\mathbf{a} = a\,\mathbf{e}_x$ is the position vector of the center of the Earth. To first-order in r/a, this equation reduces to

$$\ddot{\mathbf{r}} = \omega^2\,[(a+3\,x)\,\mathbf{e}_x - \mathbf{r}] - \omega^2\,(a\,\mathbf{e}_x - \mathbf{r}) - 2\,\omega\,\mathbf{e}_z\times\dot{\mathbf{r}}, \tag{47}$$

in the x-y plane, where use has been made of (45). Hence,

$$\ddot{\mathbf{r}} = 3\,\omega^2\,x\,\mathbf{e}_x - 2\,\omega\,\mathbf{e}_z\times\dot{\mathbf{r}}, \tag{48}$$

which yields

$$\ddot{x} = 3\,\omega^2\,x + 2\,\omega\,\dot{y}, \tag{49}$$

$$\ddot{y} = -2\,\omega\,\dot{x}. \tag{50}$$

Let us search for a solution of the form

$$x = x_0\,\cos\omega\,t, \tag{51}$$

$$y = y_0\,\sin\omega\,t. \tag{52}$$

Substitution into (49) and (50) yields $y_0 = -2\,x_0$. Hence, the solution is

$$x = x_0\,\cos\omega\,t, \tag{53}$$

$$y = -2\,x_0\,\sin\omega\,t, \tag{54}$$

where x_0 is arbitrary. This is an elliptical orbit, centered on the satellite, whose major axis in the y-direction is twice that in the x-direction, and which orbits in the opposite sense to the satellite (*i.e.*, the orbital angular momentum is in the minus z direction).

8.1 4. (a) Let the plate lie in the x-y plane such that its long sides run parallel to the x-axis. Let the origin of the coordinate system lie at the centroid (which is also the center of mass). The mass per unit area of the plate is $m/(2\,a^2)$. Thus,

$$I_{xx} = \frac{m}{2\,a^2}\int_{-a}^{a} dx \int_{-a/2}^{a/2} y^2\,dy = \frac{1}{12}\,m\,a^2, \tag{55}$$

$$I_{yy} = \frac{m}{2\,a^2}\int_{-a}^{a} x^2\,dx \int_{-a/2}^{a/2} dy = \frac{1}{3}\,m\,a^2, \tag{56}$$

$$I_{xy} = -\frac{m}{2\,a^2}\int_{-a}^{a} x\,dx \int_{-a/2}^{a/2} y\,dy = 0. \tag{57}$$

Moreover, $I_{xz} = I_{yz} = 0$, by symmetry, and

$$I_{zz} = I_{xx} + I_{yy} = \frac{5}{12}\,m\,a^2, \tag{58}$$

by the perpendicular axis theorem. Since all of the products of inertia are zero, the x, y, z axes are the principle axes of rotation, and the associated principle moments of inertia are $(1/12)\,m\,a^2$, $(1/3)\,m\,a^2$, and $(5/12)\,m\,a^2$, respectively.

(b) Let the origin of the coordinate system lie at a corner. In this case,

$$I_{xx} = \frac{m}{2\,a^2}\int_{0}^{2a} dx \int_{0}^{a} y^2\,dy = \frac{1}{3}\,m\,a^2, \tag{59}$$

$$I_{yy} = \frac{m}{2\,a^2}\int_{0}^{2a} x^2\,dx \int_{0}^{a} dy = \frac{4}{3}\,m\,a^2, \tag{60}$$

$$I_{xy} = -\frac{m}{2\,a^2}\int_{0}^{2a} x\,dx \int_{0}^{a} y\,dy = -\frac{1}{2}\,m\,a^2. \tag{61}$$

As before, $I_{xz} = I_{yz} = 0$, by symmetry, and

$$I_{zz} = I_{xx} + I_{yy} = \frac{5}{3}\,m\,a^2, \tag{62}$$

by the perpendicular axis theorem. The fact that I_{xz} and I_{yz} are both zero indicates that the z-axis is a principle axis of rotation with the associate principle moment of inertia $(5/3)\, m\, a^2$. The fact that $I_{xy} \neq 0$ indicates that the x and y axes are not principle axes of rotation. In order to find the principle axes in the x-y plane we need to solve

$$\begin{pmatrix} I_{xx} - \lambda & I_{xy} \\ I_{xy} & I_{yy} - \lambda \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{63}$$

where λ is a principle moment of inertia, and α is the angle the principle axis subtends with the x-axis. The above equation reduces to

$$\begin{pmatrix} 1/3 - \hat{\lambda} & -1/2 \\ -1/2 & 4/3 - \hat{\lambda} \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{64}$$

where $\hat{\lambda} = \lambda/(m\, a^2)$. Setting the determinant of the matrix to zero, we obtain

$$\hat{\lambda}^2 - \frac{5}{3}\,\hat{\lambda} + \frac{7}{36} = 0. \tag{65}$$

The solutions are

$$\hat{\lambda} = \frac{5}{6} \pm \frac{1}{\sqrt{2}}. \tag{66}$$

Also, from the first line of (64),

$$\tan\alpha = 2\,(1/3 - \hat{\lambda}). \tag{67}$$

Hence, the two principle axes in the x-y plane are such that $\alpha = 22.5^\circ$, $\lambda = 0.126\, m\, a^2$, and $\alpha = 112.5^\circ$, $\lambda = 1.54\, m\, a^2$.

8.2 5. From the lecture notes

$$\tan\theta = \frac{I_\perp}{I_\parallel}\,\tan\alpha. \tag{68}$$

We wish to find

$$\beta = \alpha - \theta. \tag{69}$$

A standard trigonometric identity tells us that

$$\tan\beta = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\,\tan\theta}. \tag{70}$$

Thus, it follows from (68) that

$$\tan\beta = \frac{(I_{\|} - I_{\perp})\,\tan\alpha}{I_{\|} + I_{\perp}\,\tan^2\alpha}. \tag{71}$$

6. From the previous question,

$$\tan\beta = \frac{(x-1)\,\tan\alpha}{x + \tan^2\alpha}, \tag{72}$$

where $x = I_{\|}/I_{\perp}$. So

$$\frac{d\tan\beta}{dx} = \frac{\tan\alpha\,(1+\tan^2\alpha)}{(x+\tan^2\alpha)^2}. \tag{73}$$

It follows that $d\tan\beta/dx > 0$ for $0 < \alpha < \pi/2$. Hence, the maximum value of β (at fixed α) corresponds to the maximum value of x, which is 2. So,

$$\tan\beta_{max} = \frac{\tan\alpha}{2+\tan^2\alpha}. \tag{74}$$

Now,

$$\frac{d\tan\beta_{max}}{d\tan\alpha} = \frac{2-\tan^2\alpha}{(2+\tan^2\alpha)^2}. \tag{75}$$

Thus, by varying α, we obtain the maximum value of β_{max} when $\tan\alpha = \sqrt{2}$. Hence,

$$(\tan\beta_{max})_{max} = \frac{1}{\sqrt{8}}. \tag{76}$$

Hence, we conclude that

$$\beta_{max} = \tan^{-1}\left(\frac{1}{\sqrt{8}}\right) = 19.47^\circ, \tag{77}$$

$$\alpha_{max} = \tan^{-1}\sqrt{2} = 54.73^\circ. \tag{78}$$

Physics 336K: Newtonian Dynamics
Homework 7: Solutions

8.4 1. Let the z'-axis run along the rod, and let the origin lie midway along the rod. Thus, by symmetry, the x'-, y'-, and z'-axes are principle axes of rotation. Euler's equations are

$$T_{x'} = I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \tag{1}$$

$$T_{y'} = I_{y'y'}\,\dot{\omega}_{y'} - (I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \tag{2}$$

$$T_{z'} = I_{z'z'}\,\dot{\omega}_{z'} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}. \tag{3}$$

From standard mechanics, $I_{x'x'} = I_{y'y'} = (1/12)\,m\,l^2$ and $I_{z'z'} = 0$. Now, we are told that $\boldsymbol{\omega}$ is constant. Without loss of generality, we can say that $\boldsymbol{\omega}$ lies in the x'-z' plane. Hence,

$$\omega = \omega\,(\sin\alpha,\ 0,\ \cos\alpha). \tag{4}$$

Now, $\dot{\omega}_{x'} = \dot{\omega}_{y'} = \dot{\omega}_{z'} = 0$. So, Euler's equations yield

$$T_{x'} = 0, \tag{5}$$

$$T_{y'} = \frac{1}{12}\,m\,l^2\,\omega^2\,\sin\alpha\,\cos\alpha, \tag{6}$$

$$T_{z'} = 0, \tag{7}$$

or

$$\mathbf{T} = \frac{1}{12}\,m\,l^2\,\omega^2\,\sin\alpha\,\cos\alpha\,\mathbf{e}_{y'}. \tag{8}$$

Furthermore,

$$\mathbf{L} = \omega\,(I_{x'x'}\,\sin\alpha,\ 0,\ I_{z'z'}\,\cos\alpha) = \frac{1}{12}\,m\,l^2\,\omega\,\sin\alpha\,\mathbf{e}_{x'}. \tag{9}$$

Now, the rod runs along $\mathbf{e}_{z'}$. Thus, the angular momentum vector is perpendicular to the rod, and the torque is perpendicular to both the rod and the angular momentum vector. Moreover,

$$L = \frac{1}{12}\,m\,l^2\,\omega\,\sin\alpha \tag{10}$$

$$T = \frac{1}{12}\,m\,l^2\,\omega^2\,\sin\alpha\,\cos\alpha. \tag{11}$$

2. Let the z'-axis run perpendicular to the disk, and let the origin lie at the center of the disk. Thus, by symmetry, the x'-, y'-, and z'-axes are principle axes of rotation. Euler's equations are

$$T_{x'} = I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \tag{12}$$

$$T_{y'} = I_{y'y'}\,\dot{\omega}_{y'} - (I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \tag{13}$$

$$T_{z'} = I_{z'z'}\,\dot{\omega}_{z'} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}. \tag{14}$$

From standard mechanics, $I_{x'x'} = I_{y'y'} = (1/4)\,m\,a^2$ and $I_{z'z'} = (1/2)\,m\,a^2$. Now, we are told that $\boldsymbol{\omega}$ is constant. Without loss of generality, we can say that $\boldsymbol{\omega}$ lies in the x'-z' plane. Hence,

$$\omega = \omega\,(\sin\alpha,\,0,\,\cos\alpha). \tag{15}$$

Now, $\dot{\omega}_{x'} = \dot{\omega}_{y'} = \dot{\omega}_{z'} = 0$. So, Euler's equations yield

$$T_{x'} = 0, \tag{16}$$

$$T_{y'} = -\frac{1}{4}\,m\,a^2\,\omega^2\,\sin\alpha\,\cos\alpha, \tag{17}$$

$$T_{z'} = 0, \tag{18}$$

or

$$\mathbf{T} = -\frac{1}{4}\,m\,a^2\,\omega^2\,\sin\alpha\,\cos\alpha\,\mathbf{e}_{y'}. \tag{19}$$

Furthermore,

$$\mathbf{L} = \omega\,(I_{x'x'}\,\sin\alpha,\,0,\,I_{z'z'}\,\cos\alpha) = \frac{1}{4}\,m\,a^2\,\omega\,\sin\alpha\,\mathbf{e}_{x'} + \frac{1}{2}\,m\,a^2\,\omega\,\cos\,\alpha\,\mathbf{e}_{z'}. \tag{20}$$

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Exercise 8.6

An isolated rigid body feels no torques. If it has an axis of symmetry, the Euler equations can be written

$$0 = I_\perp \dot{\omega}_{x'} + (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{y'}$$

$$0 = I_\perp \dot{\omega}_{y'} - (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{x'}$$

$$0 = I_\parallel \dot{\omega}_{z'}$$

where z' is the coordinate along the symmetry axis.

The solutions for rotation around the symmetry axis are

$$\begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix}$$

where ω_0 is a constant angular frequency. A solution for rotation around a nonsymmetry axis is

$$\begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix} = \begin{pmatrix} \omega_0 \\ 0 \\ 0 \end{pmatrix}$$

We first consider small perturbations to the rotation about the symmetry axis:

$$\begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix} = \begin{pmatrix} \mu(t) \\ \lambda(t) \\ \omega_0 + \nu(t) \end{pmatrix}$$

The Euler equations to first order in μ, λ, and ν are

$$0 = I_\perp \dot{\mu} + (I_\parallel - I_\perp)\,\omega_0 \lambda$$

$$0 = I_\perp \dot{\lambda} - (I_\parallel - I_\perp)\,\omega_0 \mu$$

$$0 = I_\parallel \dot{\nu}$$

The general solution are

$$\mu(t) = \epsilon \cos(\Omega t + \phi)$$

$$\lambda(t) = \epsilon \sin(\Omega t + \phi)$$

$$\nu(t) = \nu_0$$

where $\Omega = \frac{I_\parallel - I_\perp}{I_\perp}\omega_0$ and ϵ, ϕ, and ν_0 are constants. None of the perturbations grows with time, so rotations about the symmetry axis are stable.

We now consider small perturbations to the rotation about the non-symmetry axis:

$$\begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix} = \begin{pmatrix} \omega_0 + \nu(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}$$

The Euler equations to first order in μ, ν, λ are

$$0 = I_\perp \dot{\nu}$$

$$0 = I_\perp \dot{\mu} - (I_\parallel - I_\perp)\omega_0 \lambda$$

$$0 = I_\parallel \dot{\lambda}$$

The general solutions are

$$\nu(t) = \nu_0$$

$$\mu(t) = \mu_0 + \Omega \lambda_0 t$$

$$\lambda(t) = \lambda_0$$

where ν_0, μ_0, and λ_0 are constants. If $\lambda_0 \neq 0$, the perturbation $\mu(t)$ grows linearly with time. Thus rotations about a nonsymmetry axis are unstable.