To first-order (neglecting $\Omega^2$), the equation of lateral displacement (out of the $y$-$z$ plane) is

$$\ddot{x} = 2\,\Omega\,\sin\lambda\,\dot{y} = -2\,\Omega\,v_0\,\sin\lambda\,\cos\alpha, \tag{14}$$

where use has been made of (11). The solution that satisfies the initial conditions is

$$x = -\Omega\,v_0\,\sin\lambda\,\cos\alpha\,t^2. \tag{15}$$

The net lateral displacement $d = x(T)$ then becomes

$$d = -\frac{4\,\Omega\,v_0^3\,\sin\lambda\,\sin^2\alpha\,\cos\alpha}{g^2}, \tag{16}$$

where use has been made of (13). The minus sign indicates that the deflection is southward.

7.3 2) As before,

$$\begin{aligned} \ddot{x} &= 2\,\Omega\,\sin\lambda\,\dot{y}, && (17)\\ \ddot{y} &= -2\,\Omega\,\sin\lambda\,\dot{x} + 2\,\Omega\,\cos\lambda\,\dot{z}, && (18)\\ \ddot{z} &= -g - 2\,\Omega\,\cos\lambda\,\dot{y}. && (19) \end{aligned}$$

Suppose that the initial conditions at $t = 0$ are $x = y = z = 0$ and

$$\begin{aligned} \dot{x} &= 0, && (20)\\ \dot{y} &= 0, && (21)\\ \dot{z} &= v_0. && (22) \end{aligned}$$

To lowest order (neglecting $\Omega$), we find that

$$\begin{aligned} \ddot{x} &= 0, && (23)\\ \ddot{y} &= 0, && (24)\\ \ddot{z} &= -g. && (25) \end{aligned}$$

The solution that satisfies the initial conditions is

$$x = 0, \tag{26}$$

$$y = 0, \tag{27}$$

$$z = v_0\,t - (1/2)\,g\,t^2. \tag{28}$$

In particular, the time of flight (*i.e.*, the non-trivial root of $z = 0$) is

$$T = \frac{2\,v_0}{g}, \tag{29}$$

and the maximum height attained is

$$h = \frac{v_0^2}{2\,g}. \tag{30}$$

To first-order (neglecting $\Omega^2$), the equation of horizontal displacement is

$$\ddot{y} = 2\,\Omega\,\cos\lambda\,\dot{z} = 2\,\Omega\,\cos\lambda\,(v_0 - g\,t) \tag{31}$$

where use has been made of (28). The solution that satisfies the initial conditions is

$$y = \Omega\,\cos\lambda\left(v_0\,t^2 - \frac{1}{3}\,g\,t^3\right). \tag{32}$$

The net lateral displacement $d = x(T)$ then becomes

$$d = \frac{4}{3}\,\frac{\Omega\,v_0^3\,\cos\lambda}{g^2}. \tag{33}$$

where use has been made of (29)

Suppose that the initial conditions at $t = 0$ are $x = y = 0$, $z = h$, and $\dot{x} = \dot{y} = \dot{z} = 0$. To lowest order (neglecting $\Omega$), we find that

$$\ddot{x} = 0, \tag{34}$$

$$\ddot{y} = 0, \tag{35}$$

$$\ddot{z} = -g. \tag{36}$$

The solution that satisfies the initial conditions is

$$\begin{aligned} x &= 0, & (37) \\ y &= 0, & (38) \\ z &= h - (1/2)\,g\,t^2. & (39) \end{aligned}$$

The time of flight (*i.e.*, the non-trivial root of $z = 0$) is

$$T = \left(\frac{2\,h}{g}\right)^{1/2} = \frac{v_0}{g}, \tag{40}$$

where use has been made of (30). To first-order (neglecting $\Omega^2$), the equation of horizontal displacement is

$$\ddot{y} = 2\,\Omega\,\cos\lambda\,\dot{z} = -2\,\Omega\,g\,\cos\lambda\,t, \tag{41}$$

where use has been made of (39). The solution that satisfies the initial conditions is

$$x = -\frac{1}{3}\,\Omega\,g\,\cos\lambda\,t^3. \tag{42}$$

The net lateral displacement $d = x(T)$ then becomes

$$d = -\frac{1}{3}\,\frac{\Omega\,v_0^3\,\cos\lambda}{g^2}, \tag{43}$$

where use has been made of (40). Note that this displacement is four times less in magnitude, and in the opposite direction, to the displacement (33).

7.6 3. The satellite's angular velocity is

$$\boldsymbol{\omega} = \omega\,\mathbf{e}_z, \tag{44}$$

where

$$\omega = \left(\frac{G\,M}{a^3}\right)^{1/2}, \tag{45}$$

and $M$ is the mass of the Earth. The equation of motion of the mass in a frame co-moving with the satellite is

$$\ddot{\mathbf{r}} = -G\,M\,\frac{(\mathbf{r}-\mathbf{a})}{(r^2 - 2\,\mathbf{r}\cdot\mathbf{a} + a^2)^{3/2}} - \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}-\mathbf{a})] - 2\,\boldsymbol{\omega}\times\dot{\mathbf{r}}, \tag{46}$$

where $\mathbf{a} = a\,\mathbf{e}_x$ is the position vector of the center of the Earth. To first-order in $r/a$, this equation reduces to

$$\ddot{\mathbf{r}} = \omega^2\,[(a + 3\,x)\,\mathbf{e}_x - \mathbf{r}] - \omega^2\,(a\,\mathbf{e}_x - \mathbf{r}) - 2\,\omega\,\mathbf{e}_z\times\dot{\mathbf{r}}, \tag{47}$$

in the $x$-$y$ plane, where use has been made of (45). Hence,

$$\ddot{\mathbf{r}} = 3\,\omega^2\,x\,\mathbf{e}_x - 2\,\omega\,\mathbf{e}_z\times\dot{\mathbf{r}}, \tag{48}$$

which yields

$$\ddot{x} = 3\,\omega^2\,x + 2\,\omega\,\dot{y}, \tag{49}$$

$$\ddot{y} = -2\,\omega\,\dot{x}. \tag{50}$$

Let us search for a solution of the form

$$x = x_0\,\cos\omega\,t, \tag{51}$$

$$y = y_0\,\sin\omega\,t. \tag{52}$$

Substitution into (49) and (50) yields $y_0 = -2\,x_0$. Hence, the solution is

$$x = x_0\,\cos\omega\,t, \tag{53}$$

$$y = -2\,x_0\,\sin\omega\,t, \tag{54}$$

where $x_0$ is arbitrary. This is an elliptical orbit, centered on the satellite, whose major axis in the $y$-direction is twice that in the $x$-direction, and which orbits in the opposite sense to the satellite (*i.e.*, the orbital angular momentum is in the minus $z$ direction).

4. (a) Let the plate lie in the $x$-$y$ plane such that its long sides run parallel to the $x$-axis. Let the origin of the coordinate system lie at the centroid

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## Exercise 7.4

The angular frequency vector is $\vec{\Omega} = \Omega\hat{z}$
The origin in the corotating frame is $\vec{R} = R\hat{x}$
The equations of motion for the projectile in the corotating frame are

$$\ddot{\vec{r}} = -g\hat{z} - \vec{\Omega}\times\left[\vec{\Omega}\times(\vec{R}+\vec{r})\right] - 2\vec{\Omega}\times\dot{\vec{r}}$$

The position vector is $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

The cross products in the equations of motion are

$$\vec{\Omega}\times\left[\vec{\Omega}\times(\vec{R}+\vec{r})\right] = \vec{\Omega}\cdot(\vec{R}+\vec{r})\,\vec{\Omega} - \Omega^2(\vec{R}+\vec{r})$$

$$= \Omega z\begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} - \Omega^2\begin{pmatrix} R+x \\ y \\ z \end{pmatrix} = -\Omega^2\begin{pmatrix} R+x \\ y \\ 0 \end{pmatrix}$$

$$\vec{\Omega}\times\dot{\vec{r}} = \Omega\begin{pmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{pmatrix}$$

Thus the three component of the equations of motion are

$$\ddot{x} = \Omega^2(R+x) + 2\Omega\dot{y}$$

$$\ddot{y} = \Omega^2 y - 2\Omega\dot{x}$$

$$\ddot{z} = -g$$

The initial condition are

$x(0) = 0, \quad y(0) = 0 \quad z(0) = 0$

$\dot{x}(0) = v_0 \cos\alpha \quad \dot{y}(0) = 0 \quad \dot{z}(0) = v_0 \sin\alpha$

If $\Omega = 0$, the solution is

$x(t) = (v_0 \cos\alpha)\, t$

$y(t) = 0$

$z(t) = (v_0 \sin\alpha)\, t - \frac{1}{2} g t^2$

The time $T$ of impact, which satisfies $z(T) = 0$, is

$$T = \frac{2 v_0 \sin\alpha}{g}$$

The coordinate of the impact point are

$$x(T) = \frac{2 v_0 \sin\alpha \cos\alpha}{g}$$

$y(T) = 0$

We now consider the effects of the rotation. The Z coordinate is not affected, so the time T of impact remains the same. We write the X coordinate as

$$X(t) = (v_0 \cos\alpha)t + x(t)$$

where $x(t)$ goes to 0 as $\Omega \to 0$. The equations for $x$ and $y$ are

$$\ddot{x} = \Omega^2[R + (v_0 \cos\alpha)t + x] + 2\Omega\dot{y}$$

$$\ddot{y} = \Omega^2 y - 2\Omega[v_0 \cos\alpha + \dot{x}]$$

The consistent choice for how $x$ and $y$ scale with $\Omega$ is $x \sim \Omega^2$ and $y \sim \Omega$. Dropping terms that are higher order in $\Omega$, the equations become

$$\ddot{x} = \Omega^2[R + (v_0 \cos\alpha)t] + 2\Omega\dot{y}$$

$$\ddot{y} = -2\Omega(v_0 \cos\alpha)$$

The solution for $y$ that satisfies the initial conditions is

$$y(t) = -\Omega(v_0 \cos\alpha)t^2$$

The tangential deflection is therefore

$$y(T) = -\Omega(v_0 \cos\alpha)\left(\frac{2 v_0 \sin\alpha}{g}\right)^2$$

$$= -4\,\frac{\Omega v_0^3 \sin^2\alpha \cos\alpha}{g^2}$$

Inserting the solution for $y(t)$ into the equation for $x$, it becomes

$$\ddot{x} = \Omega^2\left[R + (v_0 \cos\alpha)t\right] + 2\Omega\left[-2\Omega(v_0 \cos\alpha)t\right]$$

$$= \Omega^2\left[R - 3(v_0 \cos\alpha)t\right]$$

The boundary conditions on $x$ are

$$x(0) = 0 \quad \dot{x}(0)$$

The resulting solution is

$$x(t) = \Omega^2\left[R \cdot \tfrac{1}{2}t^2 - 3(v_0 \cos\alpha)\tfrac{1}{6}t^3\right]$$

$$= \tfrac{1}{2}\Omega^2\left[Rt^2 - (v_0 \cos\alpha)t^3\right]$$

The radial displacement is

$$x(T) = \tfrac{1}{2}\Omega^2\left[R\cdot\left(\frac{2 v_0 \sin\alpha}{g}\right)^2 - (v_0 \cos\alpha)\left(\frac{2 v_0 \sin\alpha}{g}\right)^3\right]$$

$$= 2\,\frac{\Omega^2 v_0^2 \sin^2\alpha}{g^2}\left[R - \frac{2 v_0^2 \sin\alpha \cos\alpha}{g}\right]$$