Fitzpatrick, Chapter 6

Exercise 6.3

The total orbital angular momentum is

$$\vec{L} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

We can express the position vectors as linear combinations of the center-of-mass position $\vec{R}$ and the separation vector $\vec{r} = \vec{r}_2 - \vec{r}_1$:

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r}$$

The total angular momentum then becomes

$$\vec{L} = m_1\left(\vec{R} - \frac{m_2}{m_1+m_2}\vec{r}\right) \times \left(\dot{\vec{R}} - \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)$$

$$+ m_2\left(\vec{R} + \frac{m_1}{m_1+m_2}\vec{r}\right) \times \left(\dot{\vec{R}} + \frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)$$

The $\vec{r} \times \dot{\vec{R}}$ terms cancel and so do the $\vec{R} \times \dot{\vec{r}}$ terms:

$$\vec{L} = (m_1+m_2)\vec{R} \times \dot{\vec{R}} + \left(\frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_1^2 m_2}{(m_1+m_2)^2}\right)\vec{r} \times \dot{\vec{r}}$$

$$= M \cdot \vec{R} \times \dot{\vec{R}} + \mu \vec{r} \times \dot{\vec{r}}$$

where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$.

Note that $\mathcal{E}_n = \mathcal{E}$. From the previous problem,

$$\frac{d\sigma}{d\mathcal{E}_n'} = \frac{(m_n + m_p)^2}{m_n\, m_p}\,\frac{\pi}{\mathcal{E}}\,\frac{d\sigma}{d\Omega} \simeq \frac{4\pi}{\mathcal{E}}\,\frac{d\sigma}{d\Omega}. \tag{32}$$

Furthermore,

$$\mathcal{E} = \left(\frac{m_n + m_p}{m_p}\right) E \simeq 2\,E, \tag{33}$$

which implies that

$$\frac{d\sigma}{d\Omega} \simeq \frac{E}{2\pi}\,\frac{d\sigma}{d\mathcal{E}_n'}. \tag{34}$$

It follows that $d\sigma/d\Omega$ is constant for scattering angles in the rangle $\theta_{\rm min}$ to $\theta_{\rm max}$, where $\theta_{\rm min}$ is the scattering angle at which $\mathcal{E}_n' = \mathcal{E}$, and $\theta_{\rm max}$ the scattering angle at which $\mathcal{E}_n' = 0$. Outside this range, the scattering cross-section is zero. Now,

$$\mathcal{E}_n' = \cos^2\psi\,\mathcal{E} = \cos^2(\theta/2)\,\mathcal{E} \tag{35}$$

for equal mass scattering. So, $\theta_{\rm min} = 0$ and $\theta_{\rm max} = \pi$. Hence, the differential scattering cross-section in the center of mass frame is uniform over all scattering angles.

6. From the lecture notes,

$$\theta = \pi - 2\int_0^{u_{\rm max}} \frac{b\,du}{\sqrt{1 - b^2\,u^2 - U(u)/E}}. \tag{36}$$

Suppose that $U(u) = k\,u^2$. It follows that

$$\theta = \pi - 2\int_0^{u_{\rm max}} \frac{b\,du}{\sqrt{1 - (b^2 + k/E)\,u^2}}, \tag{37}$$

where $u_{\rm max} = 1/\sqrt{b^2 + k/E}$. Let $y = u/u_{\rm max}$. We obtain

$$\theta = \pi - 2\,\frac{b}{\sqrt{b^2 + k/E}}\int_0^1 \frac{dy}{\sqrt{1 - y^2}} = \pi - \frac{b}{\sqrt{b^2 + k/E}}\,\pi, \tag{38}$$

which can be rearranged to give

$$b^2 = \frac{k\,\pi}{E}\,\frac{(1-\theta/\pi)^2}{\theta\,(2-\theta/\pi)}. \tag{39}$$

Hence,

$$2\,b\,\frac{db}{d\theta} = -\frac{2\,k}{\pi\,E}\,\frac{(1-\theta/\pi)}{(\theta/\pi)^2\,(2-\theta/\pi)^2}. \tag{40}$$

Now,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|, \tag{41}$$

yielding

$$\frac{d\sigma}{d\Omega} = \frac{k}{\pi\,E}\,\frac{1}{\sin\theta}\,\frac{(1-\theta/\pi)}{(\theta/\pi)^2\,(2-\theta/\pi)^2}. \tag{42}$$