The orbital speed at perihelion is (since the radial component of the velocity is zero)

$$v_p = r_p\,\dot{\theta} = \frac{h}{r_p} = \frac{h}{a\,(1-e)} = 5.43\times 10^4\,\mathrm{m\,s^{-1}} = 54.3\,\mathrm{km\,s^{-1}}. \tag{9}$$

Likewise, the orbital speed at aphelion is

$$v_a = r_a\,\dot{\theta} = \frac{h}{r_a} = \frac{h}{a\,(1+e)} = 9.12\times 10^2\,\mathrm{m\,s^{-1}} = 0.912\,\mathrm{km\,s^{-1}}. \tag{10}$$

5. 2. The conserved energy per unit mass of the comet is

$$\mathcal{E} = \frac{v^2}{2} - \frac{G\,M}{r}. \tag{11}$$

Now, the Earth is in an approximately circular orbit of radius r_0 (where $r_0 = 1\,\mathrm{AU}$). Its orbital velocity is

$$v_0 = \left(\frac{G\,M}{r_0}\right)^{1/2}. \tag{12}$$

Let $d = r/r_0$ and $q = v/v_0$. It follows that

$$\mathcal{E} = \frac{v_0^2}{2\,d}\left(q^2\,d - 2\right). \tag{13}$$

Now, the comet's orbit is elliptical/parabolic/hyperbolic depending on whether $\mathcal{E} < 0\,/\,\mathcal{E} = 0\,/\,\mathcal{E} > 0$. Since, $v_0^2/2\,d$ is positive, it follows that the orbit is elliptical/parabolic/hyperbolic depending on whether $q^2\,d < 2\,/\,q^2\,d = 0\,/\,q^2\,d > 0$.

3. A Keplerian orbit is characterized by

$$r = \frac{a\,(1-e^2)}{1+e\,\cos\theta}, \tag{14}$$

$$r^2\,\dot{\theta} = h. \tag{15}$$

Let $r = a + \delta r$, where $\delta r \sim \mathcal{O}(e)$. Ignoring terms of $\mathcal{O}(e^2)$, the above expression yields $\delta r \simeq -e\,a\,\cos\theta$, so that

$$r \simeq a\,(1 - e\,\cos\theta)\,. \tag{19}$$

However, this is the same as (16). Applying the sine rule to $\triangle SEC$, we obtain $\sin SEQ/SQ = \sin SQE/SE$, or $\sin(\theta - \alpha)/(2\,e\,a) = \sin\alpha/r$. Let $\theta = \alpha + \delta\theta$, where $\delta\theta \sim \mathcal{O}(e)$. Neglecting terms of $\mathcal{O}(e^2)$, we obtain $\delta\theta \simeq 2\,e\,\sin\theta$. Hence,

$$\alpha \simeq \theta - 2\,e\,\sin\theta, \tag{20}$$

and so,

$$\dot{\alpha} \simeq \dot{\theta}\,(1 - 2\,e\,\cos\theta)\,. \tag{21}$$

However, from (17), the right-hand side of the above equation is constant (to first-order in e). Hence, $\dot{\alpha}$ is constant (to first-order in e). We conclude that a Ptolemaic orbit in which α increases uniformly in time reproduces a Keplerian orbit (to first-order in e).

5. 4. The Earth's orbit about the Sun is characterized by

$$\begin{aligned} r &= a\,(1 - e\,\cos E), & (22)\\ E - e\,\sin E &= 2\pi\left(\frac{t}{T}\right), & (23)\\ \tan(\theta/2) &= \left(\frac{1+e}{1-e}\right)^{1/2}\tan(E/2), & (24) \end{aligned}$$

where E is the elliptic anomaly, $e = 0.01673$, and $T = 365.24$ days. At $t = 0$, is is easily seen that $E = 0$, $\theta = 0$, and $r = a\,(1-e)$, which corresponds to the perihelion point. After the Earth's radius vector has rotated 90°, we have $\theta = \pi/2$. Thus, from (24),

$$\tan(E/2) = \left(\frac{1-e}{1+e}\right)^{1/2}\tan(\pi/4) = 0.9834. \tag{25}$$

It follows that $E = 0.4947\,\pi$. So, $E - e\,\sin E = 0.4893\,\pi$. Hence, from (23),

$$t = \frac{0.4893\,\pi}{2\pi}\,T = 89.37\,\text{days}. \tag{26}$$

This is the time interval for the Earth's radius vector to rotate through 90°, starting from the perihelion point.

At the aphelion point, $\theta = \pi$, and it is easily seen that $E = \pi$ and $E - e\,\sin E = \pi$. It follows that

$$t = \frac{\pi}{2\pi}\,T = 182.62\,\text{days}. \tag{27}$$

After the Earth's radius vector has rotated 90°, we have $\theta = 3\pi/2$. Thus, from (24),

$$\tan(E/2) = \left(\frac{1-e}{1+e}\right)^{1/2}\tan(3\pi/4) = -0.9834. \tag{28}$$

It follows that $E = 1.5053\,\pi$. So, $E - e\,\sin E = 1.5106\,\pi$. Hence, from (23),

$$t = \frac{1.5106\,\pi}{2\pi}\,T = 275.86\,\text{days}. \tag{29}$$

Thus, the time interval for the Earth's radius vector to rotate through 90°, starting from the aphelion point, is $275.86 - 182.62 = 93.25$ days.

5. The equation of the parabolic ($e = 1$) orbit of a comet with perihelion distance p (at $\theta = 0$) is

$$r = \frac{2\,p}{1+\cos\theta}. \tag{30}$$

The equation of the Earth's circular orbit of radius a is

$$r = a. \tag{31}$$

These two orbits intersect when

$$a = \frac{2\,p}{1+\cos\theta}, \tag{32}$$

which can be rearranged to give

$$\cos\theta = -1 + \frac{2\,p}{a}. \tag{33}$$

Physics 336K: Newtonian Dynamics
Homework 4: Solutions

1. A particle moving in a central force field whose potential energy per unit mass is $-k/r^2$ (which would produce a central force varying as $1/r^3$) has a conserved energy per unit mass

$$\mathcal{E} = \frac{1}{2}\,\dot{r}^{\,2} + \frac{1}{2}\,\frac{h^2}{r^2} - \frac{k}{r^2}, \tag{1}$$

where h is the conserved angular momentum per unit mass. Hence,

$$r^2\,\dot{r}^{\,2} - 2\,\mathcal{E}\,r^2 = \mathcal{C}, \tag{2}$$

where $\mathcal{C} = 2\,k - h^2$ is a constant. Let $u = r^2$. It follows that $\dot{u} = 2\,r\,\dot{r}$. Thus,

$$\dot{u}^{\,2} - 8\,\mathcal{E}\,u = 4\,\mathcal{C}. \tag{3}$$

Differentiation with respect to time yields

$$\dot{u}\,(\ddot{u} - 4\,\mathcal{E}) = 0. \tag{4}$$

Thus, either $\dot{u} = 0$, which corresponds to an (unstable) circular orbit, or

$$\ddot{u} = 4\,\mathcal{E}. \tag{5}$$

The most general solution of the above equation is

$$u = r^2 = 2\,\mathcal{E}\,t^2 + B\,t + C, \tag{6}$$

where B and C are arbitrary constants.

5.9 ~~2~~. The equation of motion of a point mass moving in a central potential (per unit mass) $V(r)$ is

$$\frac{d^2u}{d\theta^2} + u = -h^{-2}\,\frac{dV}{du}, \tag{7}$$

where $u = r^{-1}$, and r, θ are polar coordinates. Here, h is the angular momentum per unit mass.

Suppose that

$$r = r_0 \cos\theta, \tag{8}$$

which is the equation of a circular orbit which passes through the origin. It follows that

$$u = \frac{r_0^{-1}}{\cos\theta}, \tag{9}$$

and

$$\frac{d^2u}{d\theta^2} = \frac{2\,r_0^{-1}}{\cos^3\theta} - \frac{r_0^{-1}}{\cos\theta} = 2\,r_0^{\,2}\,u^3 - u. \tag{10}$$

So,

$$-h^{-1}\,\frac{dV}{du} = \frac{d^2u}{d\theta^2} + u = 2\,r_0^{\,2}\,u^3 - u + u = 2\,r_0^{\,2}\,u^3, \tag{11}$$

which can be integrated to give

$$V = -\frac{h^2\,r_0^{\,2}}{2}\,u^4 = -\frac{h^2\,r_0^{\,2}}{2\,r^4}. \tag{12}$$

The central force per unit mass is thus

$$f = -\frac{dV}{dr} = -\frac{2\,h^2\,r_0^{\,2}}{r^5}. \tag{13}$$

Thus, the force law is inverse-fifth.

3. Let

$$r = r_0\,\mathrm{e}^{\,k\,\theta}, \tag{14}$$

which corresponds to an expanding spiral orbit. So,

$$u = r_0^{-1}\,\mathrm{e}^{-k\,\theta}, \tag{15}$$

and

$$\frac{d^2u}{d\theta^2} = k^2\,r_0^{-1}\,\mathrm{e}^{-k\,\theta} = k^2\,u^2. \tag{16}$$

Thus, (7) yields

$$\frac{dV}{du} = -h^2\,(1+k^2)\,u, \tag{17}$$

which can be integrated to give

$$V = -\frac{h^2\,(1+k^2)}{2}\,u^2 = -\frac{h^2\,(1+k^2)}{2\,r^2}. \tag{18}$$

Hence,

$$f = -\frac{dV}{dr} = -\frac{h^2\,(1+k^2)}{r^3}, \tag{19}$$

and the force law is inverse-third.

In the potential (18), the radial equation of motion, (7), yields

$$\frac{d^2u}{d\theta^2} + u = (1+k^2)\,u, \tag{20}$$

or

$$\frac{d^2u}{d\theta^2} + k^2\,u = 0. \tag{21}$$

We have already seen the orbit associated with the solution $r_0^{-1}\,\mathrm{e}^{-k\,\theta}$ of the above equation. However, $r_0^{-1}\,e^{\,k\,\theta}$ is also a solution. The corresponding orbit

$$r = r_0\,\mathrm{e}^{-k\,\theta} \tag{22}$$

is a decaying spiral. Finally, if $k=0$ then $u = r_0^{-1}$, and

$$r = r_0, \tag{23}$$

which corresponds to a circular orbit.

5,12 $\cancel{4}$. We are told that

$$f(r) = -c\,\frac{\mathrm{e}^{-r/a}}{r^2}, \tag{24}$$

where $c>0$ and $a>0$. A circular orbit of radius r_0 is stable provided that

$$f(r_0) + \frac{r_0}{3}\,f'(r_0) < 0. \tag{25}$$

The stability criterion yields

$$-c\,\frac{\mathrm{e}^{-r_0/a}}{r_0^2}+\frac{r_0}{3}\left(\frac{c\,\mathrm{e}^{-r_0/a}}{a\,r_0^2}+\frac{2\,c\,\mathrm{e}^{-r_0/a}}{r_0^3}\right)<0, \tag{26}$$

or

$$-c\,\frac{\mathrm{e}^{-r_0/a}}{3\,r_0^2}\left(1-\frac{r_0}{a}\right)<0. \tag{27}$$

Given that $c>0$ and $a>0$, we conclude that the orbit is stable provided that $r_0<a$.

5. A dust cloud of uniform mass density ρ generates a central force per unit mass given by Gauss' law:

$$f(r)\,S(r)=-G\,\rho\,V(r), \tag{28}$$

where $S(r)=4\pi\,r^2$ is the area of a spherical Gaussian surface of radius r, and $V(r)=(4/3)\,\pi\,r^3$ is the enclosed volume. Thus,

$$f(r)=-\frac{1}{3}\,G\,\rho\,r. \tag{29}$$

So, the total radial force per unit mass due to both the Sun and the dust cloud is

$$f(r)=-\frac{G\,M}{r}-\frac{1}{3}\,G\,\rho\,r, \tag{30}$$

where M is the solar mass. Now, the apsidal angle for a nearly circular orbit of radius r is

$$\psi=\pi\left(3+\frac{r}{f}\frac{df}{dr}\right)^{-1/2}. \tag{31}$$

However,

$$\frac{r}{f}\frac{df}{dr}=\frac{2\,G\,M/r^2-(1/3)\,G\,\rho\,r}{-G\,M/r^2-(1/3)\,G\,\rho\,r}=-2\left[\frac{1-(1/6)\,\rho\,r^3/M}{1+(1/3)\,\rho\,r^3/M}\right]. \tag{32}$$

Assuming that $\rho\,r^3/M\ll 1$, we obtain

$$\frac{r}{f}\frac{df}{dr}\simeq -2\left(1-\frac{1}{2}\,\frac{\rho\,r^3}{M}\right). \tag{33}$$

Thus, the apsidal angle is

$$\psi \simeq \pi \left(1 + \frac{\rho\, r^3}{M}\right)^{-1/2} \simeq \pi - \frac{\pi\, \rho\, r^3}{2\, M} \simeq \pi - \frac{3}{8}\, \frac{M_0}{M}, \tag{34}$$

where $M_0 = (4/3)\, \pi\, \rho\, r^3$ is the mass of dust enclosed by the planetary orbit.

5,15

~~6.~~ We are told that

$$V(r) = -\frac{G\, M}{r} - \frac{G\, M\, \epsilon}{r^3}, \tag{35}$$

where

$$\epsilon = \frac{2}{5}\, R\, \Delta R. \tag{36}$$

So, given that $f = -dV/dr$, the radial force per unit mass is

$$f = -\frac{G\, M}{r^2} - \frac{3\, G\, M\, \epsilon}{r^4}. \tag{37}$$

It follows that

$$\frac{r}{f}\, \frac{df}{dr} = \frac{2\, G\, M/r^2 + 12\, G\, M\, \epsilon/r^4}{-G\, M/r^2 - 3} = -2\, \frac{1 + 6\, \epsilon/r^2}{1 + 3\, \epsilon/r^2}. \tag{38}$$

Assuming that $\epsilon/r^2 \ll 1$, we obtain

$$\frac{r}{f}\, \frac{df}{dr} \simeq -2 - \frac{6\, \epsilon}{r^2}. \tag{39}$$

Hence, from (31), the apsidal angle is

$$\psi \simeq \pi \left(1 - \frac{6\, \epsilon}{r^2}\right)^{-1/2} \simeq \pi + \frac{3\pi\, \epsilon}{r^2} \simeq \pi + \delta\psi, \tag{40}$$

where

$$\delta\psi = \frac{6\pi}{5}\, \frac{R\, \Delta R}{r^2}. \tag{41}$$

The perihelion precession rate is

$$\dot{\phi} = \frac{2\, \delta\psi}{T}, \tag{42}$$

where T is the orbital period. But,

$$T = 2\pi\,\frac{r^{3/2}}{(G\,M)^{1/2}}. \tag{43}$$

Hence,

$$\dot{\phi} = \frac{6\,(G\,M)^{1/2}}{5\,R^{3/2}}\,\frac{\Delta R}{R}\left(\frac{R}{r}\right)^{7/2}. \tag{44}$$

Given that $G = 6.67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$, $M = 5.97\times 10^{24}\,\mathrm{kg}$, $R = 6437\,\mathrm{km}$, and $\Delta R/R = 13/4000$, we obtain

$$\dot{\phi} = 4.8\times 10^{-6}\left(\frac{R}{r}\right)^{7/2}\ \text{rad./sec} = 23.8\left(\frac{R}{r}\right)^{7/2}\ \text{deg./day}. \tag{45}$$