Let $u = (b/a)\,v^2$. The above expresion transforms to

$$\frac{1}{2\,b}\int_0^{(b/a)\,v_0^2}\frac{du}{1+u} = x_0. \tag{8}$$

This is a standard integral which can be evaluated to give

$$x_0 = \frac{1}{2\,b}\ln\left[1 + (b/a)\,v_0^2\right]. \tag{9}$$

3. 2. The gravitational potential energy per unit mass of a projectile located a height z above the surface of the Earth is

$$\mathcal{U} = -\frac{G\,M}{R+z}, \tag{10}$$

where M and R are the Earth's mass and radius, respectively. Since gravity is a conservative force, the projectile's energy per unit mass,

$$\mathcal{E} = \frac{1}{2}\,v^2 + \mathcal{U} = \frac{1}{2}\,v^2 - \frac{G\,M}{R+z}, \tag{11}$$

is a constant of the motion. Here, v is the instantaneous velocity of the projectile. Suppose that the projectile is launched vertically upward from the Earth's surface ($z = 0$) with an initial velocity v_0 such that it attains a maximum height h (when $v = 0$) above the surface. It follows from energy conservation that

$$\frac{1}{2}\,v_0^2 - \frac{G\,M}{R} = -\frac{G\,M}{R+h}. \tag{12}$$

The above expression can be solved for h to give

$$h = \frac{v_0^2/(2\,g)}{1 - v_0^2/(2\,g\,R)}, \tag{13}$$

where $g = G\,M/R^2$ is the gravitational acceleration at the Earth's surface. Neglecting the variation of the Earth's gravitational acceleration with height is equivalent to taking the limit $R \to \infty$. In this limit, the above expression yields the approximate result

$$h_0 \simeq \frac{v_0^2}{2\,g}. \tag{14}$$

Hence, (13) becomes

$$h = \frac{h_0}{1 - h_0/R}, \tag{15}$$

which implies that

$$\Delta h = h_0 - h = h_0 \left(\frac{1}{1 - h_0/R} - 1 \right) = \frac{h_0^2}{R - h_0}. \tag{16}$$

3. The block's equation of motion is

$$m \frac{dv}{dt} = -c\, v^n, \tag{17}$$

which yields

$$\frac{dv}{v^n} = -\frac{c}{m}\, dt. \tag{18}$$

The above expression can be integrated, assuming that $v = v_0$ at $t = 0$, to give

$$\int_{v_0}^{v} \frac{dv'}{v'^{\,n}} = -\frac{c}{m}\, t. \tag{19}$$

Hence,

$$\left[\frac{1}{-(n-1)\, v'^{\,n-1}} \right]_{v_0}^{v} = \frac{c}{m}\, t, \tag{20}$$

assuming that $n \neq 1$, which yields

$$\frac{1}{v^{\,n-1}} - \frac{1}{v_0^{n-1}} = (n-1)\, \frac{c}{m}\, t, \tag{21}$$

or

$$v = \left[\frac{1}{v_0^{n-1}} + (n-1)\, \frac{c}{m}\, t \right]^{-1/(n-1)}. \tag{22}$$

Now, $v = dx/dt$, so

$$dx = \left[\frac{1}{v_0^{n-1}} + (n-1)\, \frac{c}{m}\, t \right]^{-1/(n-1)} dt. \tag{23}$$

Fitzpatrick, Chapter 3

Problem 3.4

The equation of motion is

$$m\ddot{x} = -\frac{k}{x^2}$$

This can be expressed as

$$m\ddot{x} = -\frac{d}{dx}\left(-\frac{k}{x}\right)$$

Therefore the conserved energy is

$$E = \frac{1}{2}m\dot{x}^2 - \frac{k}{x}$$

The initial conditions are

$$x(0) = b$$

$$\dot{x}(0) = 0$$

Thus the energy is

$$E = -\frac{k}{b}$$

Conservation of energy implies

$$\frac{1}{2}m\dot{x}^2 - \frac{k}{x} = -\frac{k}{b}$$

We can solve this for $\dot{x}$:

$$\dot{x} = -\sqrt{\frac{2k}{m}}\sqrt{\frac{1}{x}-\frac{1}{b}}$$

This can be expressed in a form with x and t separated

$$\frac{1}{\sqrt{\frac{1}{x}-\frac{1}{b}}}\,dx = -\sqrt{\frac{2k}{m}}\,dt$$

We now integrate over the time t from 0 to τ, during which $x(t)$ changes from b to 0:

$$\int_b^0 \sqrt{\frac{bx}{b-x}}\,dx = -\sqrt{\frac{2k}{m}}\,\tau$$

The integral can be evaluated analytically:

$$\tau = \sqrt{\frac{mb}{2k}}\int_0^b dx\,\sqrt{\frac{x}{b-x}}$$

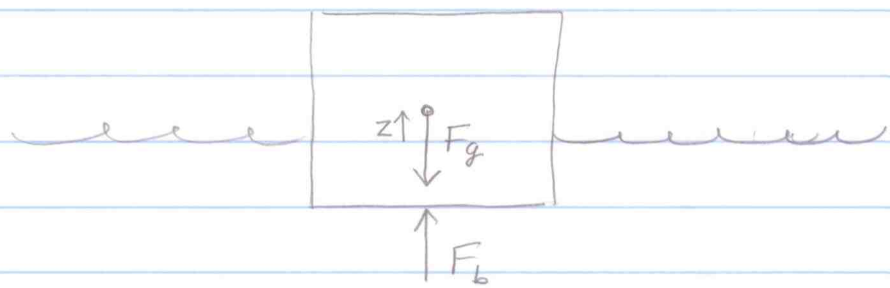
$$= \sqrt{\frac{mb}{2k}}\cdot\frac{\pi b}{2}$$

$$= \pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

Fitzpatrick, Chapter 3

Exercise 3.9

Let the body with crosssectional area A have height h and let z be the height of its center of mass relative to the height of the liquid. The volume of liquid displaced is therefore $(z - \frac{1}{2}h)A$.



The mass of the body is $\rho A h$, so the gravitational force is

$$F_g = -(\rho A h) g$$

The mass of the water displaced is $\rho_0 A(z - \frac{1}{2}h)$, so the buoyant force is

$$F_b = +\rho_0 A(z - \tfrac{1}{2}h) g$$

Newton's equation for the body is

$$(\rho A h)\ddot{z} = -(\rho A h) g + \rho_0 A (z - \tfrac{1}{2}h) g$$

At equilibrium, $z(t) = z_0$ and the equation reduces to

$$0 = -\rho h g + \rho_0 (z_0 - \tfrac{1}{2}h) g$$

The volume of water displaced is

$$V_0 = (z_0 - \tfrac{1}{2}h) A$$

$$= \frac{\rho}{\rho_0} h A$$

Newton's equation can be expressed as

$$\ddot{z} = -\frac{\rho_0 g}{\rho h}(z - z_0)$$

This is an equation for harmonic oscillations of z around z_0 with angular frequency

$$\omega = \sqrt{\frac{\rho_0 g}{\rho h}}$$

The period of the oscillations is

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi \sqrt{\frac{\rho h}{\rho_0 g}}$$

$$= 2\pi \sqrt{\frac{\rho h A}{\rho_0} \frac{1}{gA}}$$

$$= 2\pi \sqrt{\frac{V_0}{gA}}$$

Fitzpatrick, Chapter 3

Exercise 3.12

(a) The equation of motion is

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = f_0 \sin(\omega t)$$

This is the imaginary part of the equation

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = f_0 e^{i\omega t}$$

We look for a complex solution with the same frequency

$$x(t) = a\, e^{i\omega t}$$

$$\dot{x}(t) = i\omega a\, e^{i\omega t}$$

$$\ddot{x}(t) = -\omega^2 a\, e^{i\omega t}$$

The equation becomes

$$\left(-\omega^2 + 2\nu \cdot i\omega + \omega_0^2\right) a\, e^{i\omega t} = f_0 e^{i\omega t}$$

The solution for a is

$$a = \frac{f_0}{-(\omega^2 - \omega_0^2) + 2i\nu\omega}$$

$$= -\frac{f_0}{(\omega^2-\omega_0^2)^2 + 4\nu^2\omega^2}\left[(\omega^2-\omega_0^2) + 2i\nu\omega\right]$$

The corresponding real solution is

$$x(t) = \operatorname{Im}\left(a e^{i\omega t}\right)$$

$$= -\frac{f_0}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2}\left[(\omega^2-\omega_0^2)\sin(\omega t) + 2\nu\omega\cos(\omega t)\right]$$

We now consider the homogenous equation

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = 0$$

The most general solution is

$$x(t) = A e^{-\nu t}\cos(\omega_r t) + B e^{-\nu t}\sin(\omega_r t)$$

where $\omega_r = +\sqrt{\omega_0^2-\nu^2}$. Thus the most general solution of the nonhomogeneous equation is

$$\boxed{\begin{aligned} x(t) = &-\frac{f_0}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2}\left[(\omega^2-\omega_0^2)\sin(\omega t) + 2\nu\omega\cos(\omega t)\right] \\ &+ A e^{-\nu t}\cos(\omega_r t) + B e^{-\nu t}\sin(\omega_r t)\end{aligned}}$$

The solution and its first derivative at $t=0$ are

$$x(0) = -\frac{f_0\, 2\nu\omega}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2} + A$$

$$\dot{x}(0) = -\frac{f_0 \cdot 2\nu\omega^2}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2} + A(-\nu) + B\omega_r$$

The initial conditions are

$$x(0) = 0$$

$$\dot{x}(0) = v_0$$

The solutions for the amplitude A and B are

$$\boxed{\begin{aligned} A &= \frac{f_0 \cdot 2\nu\omega}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2} \\ B &= \frac{\frac{1}{\omega_r} f_0\, 2\nu\omega(\omega+\nu)}{(\omega^2-\omega_0^2)^2+4\nu^2\omega^2} + \frac{v_0}{\omega_r} \end{aligned}}$$

(b) The equation of motion is

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = f_0 \sin^2(\omega t)$$

$$= f_0 \tfrac{1}{2}\left[1 - \cos(2\omega t)\right]$$

The solution can be written

$$x(t) = \frac{f_0}{2\omega^2} + \tilde{x}(t)$$

where $\tilde{x}(t)$ is oscillatory and satisfies

$$\ddot{\tilde{x}} + 2\nu\dot{\tilde{x}} + \omega_0^2\tilde{x} = -\tfrac{1}{2} f_0 \cos(2\omega t)$$

This is the real part of the equation

$$\ddot{\tilde{x}} + 2\nu\dot{\tilde{x}} + \omega_0^2\tilde{x} = -\tfrac{1}{2}f_0 e^{2i\omega t}$$

We look for a complex solution of the form

$$\tilde{x}(t) = a e^{2i\omega t}$$

$$\dot{\tilde{x}}(t) = 2i\omega a e^{2i\omega t}$$

$$\ddot{\tilde{x}}(t) = -4\omega^2 a e^{2i\omega t}$$

The equation becomes

$$\left(-4\omega^2 + 2\nu\cdot 2i\omega + \omega_0^2\right) a e^{2i\omega t} = -\tfrac{1}{2}f_0 e^{2i\omega t}$$

The solution for a is

$$a = \frac{-\frac{1}{2}f_0}{-4\omega^2 + \omega_0^2 + 4i\nu\omega}$$

$$= \frac{\frac{1}{2}f_0}{(4\omega^2-\omega_0^2)^2 + 16\nu^2\omega^2}\left[(4\omega^2-\omega_0^2) + 4i\nu\omega\right]$$

The corresponding real solution is

$$\tilde{x}(t) = \mathrm{Re}\left(a e^{2i\omega t}\right)$$

$$= \frac{\frac{1}{2}f_0}{(4\omega^2-\omega_0^2)^2 + 16\nu^2\omega^2}\left[(4\omega^2-\omega_0^2)\cos(2\omega t) - 4\nu\omega\sin(2\omega t)\right]$$

The homogeneous equation is

$$\ddot{\tilde{x}} + 2\nu\dot{\tilde{x}} + \omega_0^2 \tilde{x} = 0$$

The general solution is

$$\tilde{x}(t) = A e^{-\nu t}\cos(\omega_r t) + B e^{-\nu t}\sin(\omega_r t)$$

where $\omega_r = \sqrt{\omega_0^2 - \nu^2}$. Thus the general solution of the nonhomogeneous equation is

$$\boxed{\begin{aligned} x(t) = \frac{f_0}{2\omega^2} + \frac{\frac{1}{2}f_0}{(4\omega^2-\omega_0^2)^2 + 16\nu^2\omega^2}\left[(4\omega^2-\omega_0^2)\cos(2\omega t) - 4\nu\omega\sin(2\omega t)\right] \\ + A e^{-\nu t}\cos(\omega_r t) + B e^{-\nu t}\sin(\omega_r t) \end{aligned}}$$

The value of x and its first time derivative at x=0 are

$$x(0) = \frac{f_0}{2\omega^2} + \frac{\frac{1}{2}f_0(4\omega^2-\omega_0^2)}{(4\omega^2-\omega_0^2)^2+16\nu^2\omega^2} + A$$

$$\dot{x}(0) = \frac{\frac{1}{2}f_0(-8\nu\omega^2)}{(4\omega^2-\omega_0^2)^2+16\nu^2\omega^2} + A(-\nu) + B\omega_r$$

The initial conditions are $x(0)=0$ and $\dot{x}(0)=v_0$
The solutions for A and B are

$$\boxed{\begin{aligned} A &= -\frac{f_0}{2\omega^2} - \frac{\frac{1}{2}f_0(4\omega^2-\omega_0^2)}{(4\omega^2-\omega_0^2)^2+16\nu^2\omega^2} \\ B &= \frac{v_0}{\omega_r} - \frac{f_0\nu}{2\omega^2\omega_r} + \frac{\frac{1}{2\omega_r}f_0\nu(4\omega^2+\omega_0^2)}{(4\omega^2-\omega_0^2)^2+16\nu^2\omega^2} \end{aligned}}$$

Equations (33) and (37) give

$$\frac{1}{1+(v_0/v_t)^2} = 1 - \left(\frac{v_f}{v_t}\right)^2, \tag{38}$$

which reduces to

$$v_f = \frac{v_0\,v_t}{(v_t^2+v_0^2)^{1/2}}. \tag{39}$$

4.1 5. The equation of motion of the electron is

$$m\,\frac{d^2\mathbf{r}}{dt^2} = -e\left(\mathbf{E} + \frac{d\mathbf{r}}{dt}\times\mathbf{B}\right), \tag{40}$$

where $\mathbf{E} = E\,\mathbf{e}_x$ and $\mathbf{B} = B\,\mathbf{e}_z$. In component form, the above equation becomes

$$\frac{d^2x}{dt^2} = -\frac{e\,B}{m}\,\frac{dy}{dt}, \tag{41}$$

$$\frac{d^2y}{dt^2} = -\frac{e\,E}{m} + \frac{e\,B}{m}\,\frac{dx}{dt}, \tag{42}$$

$$\frac{d^2z}{dt^2} = 0. \tag{43}$$

The initial conditions are $x = y = z = 0$ and $\dot{x} = v_0$ and $\dot{y} = \dot{z} = 0$ at $t = 0$. Let us try a cycloidal solution of the form

$$x = a\,\sin(\omega\,t) + b\,t, \tag{44}$$

$$y = c\,[1-\cos(\omega\,t)], \tag{45}$$

$$z = 0. \tag{46}$$

The solution satisfies the initial condition $\dot{x} = v_0$ at $t = 0$ provided

$$a\,\omega + b = v_0. \tag{47}$$

The solution automatically satisfies the other initial conditions. Substitution of the solution into (41) and (42) yields

$$-\omega^2\,a\,\sin(\omega\,t) = -c\,\omega\,\frac{e\,B}{m}\,\sin(\omega\,t), \tag{48}$$

$$\omega^2\,c\,\cos(\omega\,t) = -\frac{e\,E}{m} + \frac{e\,B}{m}\,[a\,\omega\,\cos(\omega\,t) + b]. \tag{49}$$

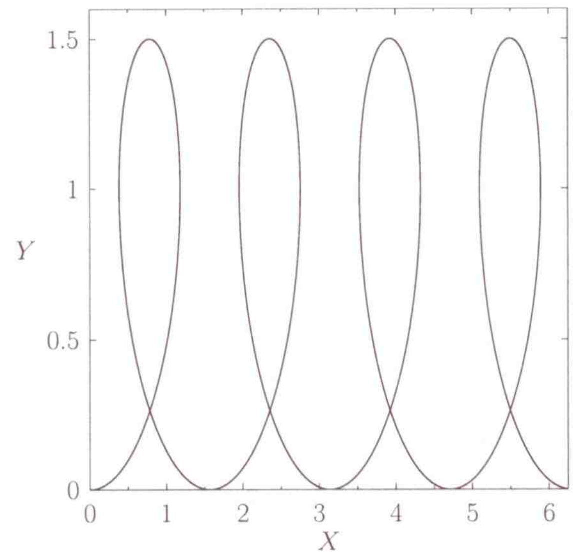


Figure 1: *Trajectory for* $\lambda = 0.25$.

In order to satisfy the previous three equations, we require

$$a = c = (v_0 - E/B)/\omega, \tag{50}$$

$$\omega = \frac{e\,B}{m}, \tag{51}$$

$$b = \frac{e}{m}\,\frac{E}{\omega} = \frac{E}{B}. \tag{52}$$

Hence, we can write

$$X = (1-\lambda)\,\sin T + \lambda\,T, \tag{53}$$

$$Y = (1-\lambda)\,(1-\cos T), \tag{54}$$

where $X = x/\rho$, $Y = y/\rho$, $\rho = v_0/\omega$, $T = \omega\,t$, and $\lambda = (E/B)/v_0$. Figure 1 shows a typical trajectory for $\lambda < 0.5$ (*i.e.*, $v_0 > 2\,E/B$). Figure 2 shows a typical trajectory for $0.5 < \lambda < 1$ (*i.e.*, $E/B < v_0 < 2\,E/B$). Figure 3 shows a typical trajectory for $\lambda > 1$ (*i.e.*, $v_0 < E/B$).

6. The x and y components of the particle's equation of motion are

$$\ddot{x} = \frac{q\,E_0}{m}\,\cos(\omega\,t), \tag{55}$$

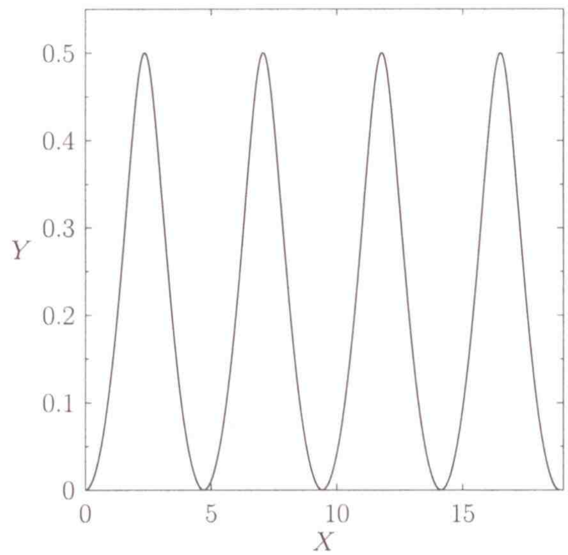


Figure 2: *Trajectory for* $\lambda = 0.75$.

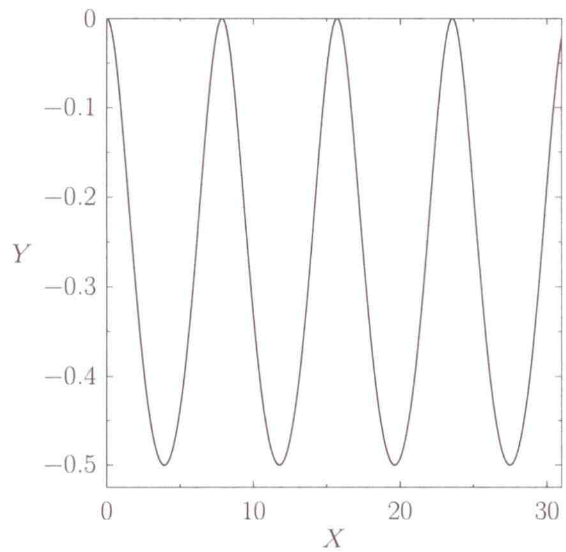


Figure 3: *Trajectory for* $\lambda = 1.25$.