Physics 336K: Newtonian Dynamics
Homework 1: Solutions

2.1. When N point particles interact via gravity the two-particle interaction force is of the form

$$\mathbf{f}_{ij} = G\, m_i\, m_j\, \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \tag{1}$$

whereas the two-particle potential energy is

$$U_{ij} = -\frac{G\, m_i\, m_j}{|\mathbf{r}_j - \mathbf{r}_1|}. \tag{2}$$

The equation of motion of the ith particle then becomes

$$m_i\, \ddot{\mathbf{r}}_i = \sum_j^{j\neq i} G\, m_i\, m_j\, \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}. \tag{3}$$

The total kinetic energy is

$$K = \frac{1}{2} \sum_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i. \tag{4}$$

The total potential energy is

$$U = \frac{1}{2} \sum_{i,j}^{i\neq j} = -\frac{1}{2} \sum_{i,j}^{i\neq j} \frac{G\, m_i\, m_j}{|\mathbf{r}_j - \mathbf{r}_i|}, \tag{5}$$

where the factor $1/2$ is to compensate for double counting. We can form the scalar product of (3) with $\dot{\mathbf{r}}_i$, and then sum over all particles, to obtain

$$\sum_i m_i\, \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i = \sum_{i,j}^{j\neq i} G\, m_i\, m_j\, \frac{\dot{\mathbf{r}}_i \cdot (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \tag{6}$$

which is equivalent to

$$\frac{1}{2}\frac{d}{dt}\sum_i m_i\,\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i = \sum_{i,j}^{j\neq i} G\,m_i\,m_j\,\frac{\dot{\mathbf{r}}_i\cdot(\mathbf{r}_j-\mathbf{r}_i)}{|\mathbf{r}_j-\mathbf{r}_i|^3}, \tag{7}$$

or

$$\frac{dK}{dt} = \sum_{i,j} G\,m_i\,m_j\,\frac{\dot{\mathbf{r}}_i\cdot(\mathbf{r}_j-\mathbf{r}_i)}{|\mathbf{r}_j-\mathbf{r}_i|^3}, \tag{8}$$

where use has been made of (4). Swapping the dummy indices i and j on the right-hand side of the above expression yields

$$\frac{dK}{dt} = \sum_{j,i}^{i\neq j} G\,m_j\,m_i\,\frac{\dot{\mathbf{r}}_j\cdot(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^3} = -\sum_{i,j}^{j\neq i} G\,m_i\,m_j\,\frac{\dot{\mathbf{r}}_j\cdot(\mathbf{r}_j-\mathbf{r}_i)}{|\mathbf{r}_j-\mathbf{r}_i|^3}. \tag{9}$$

Forming half the sum of the previous two equations, we obtain

$$\frac{dK}{dt} = \sum_{i,j}^{j\neq i} G\,m_i\,m_j\,\frac{(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_j)\cdot(\mathbf{r}_j-\mathbf{r}_i)}{|\mathbf{r}_j-\mathbf{r}_i|^3} = \frac{1}{2}\frac{d}{dt}\sum_{i,j}^{j\neq i}\frac{G\,m_i\,m_j}{|\mathbf{r}_j-\mathbf{r}_i|} = -\frac{dU}{dt}, \tag{10}$$

where use has been made of (5). Hence,

$$\frac{d(K+U)}{dt} = 0, \tag{11}$$

which implies that the total energy of the system, $E = K+U$, is a constant.

2. Suppose that

$$f(t\,x_1,\,t\,x_2,\,t\,x_3,\cdots) = t^{\alpha}\,f(x_1,\,x_2,\,x_3,\cdots) \tag{12}$$

for all t. It follows that

$$\frac{d}{dt}\,f(t\,x_1,\,t\,x_2,\,t\,x_3,\cdots) = \alpha\,t^{\alpha-1}\,f(x_1,\,x_2,\,x_3,\cdots). \tag{13}$$

But, we can also write

$$\frac{d}{dt}\,f(t\,x_1,\,t\,x_2,\,t\,x_3,\cdots) = \sum_i x_i\,\frac{\partial f(t\,x_1,\,t\,x_2,\,t\,x_3,\cdots)}{\partial(t\,x_i)}. \tag{14}$$

Hence,

$$\sum_i x_i \frac{\partial f(t\,x_1,\,t\,x_2,\,t\,x_3,\cdots)}{\partial(t\,x_i)} = \alpha\,t^{\alpha-1}\,f(x_1,\,x_2,\,x_3,\cdots). \tag{15}$$

Setting $t = 1$, we get

$$\sum_i x_i \frac{\partial f(x_1,\,x_2,\,x_3,\cdots)}{\partial x_i} = \alpha\,f(x_1,\,x_2,\,x_3,\cdots). \tag{16}$$

2.3. Now,

$$\mathbf{f}_{ij} = k_i\,k_j\,\frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^{n+1}} \tag{17}$$

is the appropriate expression for an attractive central two-particle force whose magnitude is $k_i\,k_j\,|\mathbf{r}_j - \mathbf{r}_i|^{-n}$. However,

$$k_i\,k_j\,\frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^{n+1}} = -\frac{1}{n-1}\,\frac{\partial}{\partial\mathbf{r}_i}\left(\frac{k_i\,k_j}{|\mathbf{r}_j - \mathbf{r}_i|^{n-1}}\right), \tag{18}$$

provided that $n \neq 1$. In other words,

$$\mathbf{f}_{ij} = -\frac{\partial U_{ij}}{\partial\mathbf{r}_i}, \tag{19}$$

where

$$U_{ij} = -\frac{1}{n-1}\,\frac{k_i\,k_j}{|\mathbf{r}_j - \mathbf{r}_i|^{n-1}} \tag{20}$$

is the two-particle potential energy. The total potential energy is

$$U = \frac{1}{2}\sum_{i,j}^{i\neq j} U_{ij}. \tag{21}$$

However, if $\mathbf{r}_i \to t\,\mathbf{r}_i$ for all i then $U \to U/t^{n-1}$. Hence, U is a homogeneous function of degree $1-n$. It follows, from the previous question, that

$$\sum_i \mathbf{r}_i\cdot\frac{\partial U}{\partial\mathbf{r}_i} = -(n-1)\,U. \tag{22}$$

Now, the equation of motion of the ith particle is

$$m_i\,\ddot{\mathbf{r}}_i = -\frac{\partial}{\partial \mathbf{r}_i}\sum_j^{j\neq i} U_{ij} = -\frac{1}{2}\,\frac{\partial}{\partial \mathbf{r}_i}\sum_{k,j}^{k\neq j} U_{kj}. \tag{23}$$

The final step follows because only those terms in the sum for which either $j = i$ or $k = i$ survive partial differentiation with respect to $\mathbf{r}_i$, and also because $U_{ij} = U_{ji}$. Hence, using (21), the equation of motion becomes

$$m_i\,\ddot{\mathbf{r}}_i = -\frac{\partial U}{\partial \mathbf{r}_i}. \tag{24}$$

Forming the scalar product with $\mathbf{r}_i$, and summing over all i, we get

$$\sum_i m_i\,\dot{\mathbf{r}}_i\cdot\ddot{\mathbf{r}}_i = -\sum_i \mathbf{r}_i\cdot\frac{\partial U}{\partial \mathbf{r}_i} = (n-1)\,U, \tag{25}$$

where use has been made of (22). Now,

$$\sum_i m_i\,\dot{\mathbf{r}}_i\cdot\ddot{\mathbf{r}}_i = \frac{1}{2}\,\frac{d^2}{dt^2}\sum_i m_i\,\mathbf{r}_i\cdot\mathbf{r}_i - \sum_i m_i\,\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i, \tag{26}$$

which yields

$$\sum_i m_i\,\dot{\mathbf{r}}_i\cdot\ddot{\mathbf{r}}_i = \frac{1}{2}\,\ddot{I} - 2\,K. \tag{27}$$

So, from (25),

$$\frac{1}{2}\,\ddot{I} = 2\,K + (n-1)\,U. \tag{28}$$

Now, in a steady state, $\ddot{I} = 0$, so that

$$U = -\frac{2\,K}{n-1}. \tag{29}$$

In this case, the total energy becomes

$$E = K + U = \left(\frac{n-3}{n-1}\right)K. \tag{30}$$

We need $E < 0$ for a bound state (assuming $n > 1$, so that $U \to 0$ as the system disperses to infinity). Hence, we need

$$n < 3. \tag{31}$$

In other words, there are no bound steady states for $n > 3$.

4. The virial equation, (28), for a gravitationally bound (*i.e.*, $n = 2$) system, can be written

$$\frac{1}{2}\ddot{I} = 2\,K + U = 2\,(K + U) - U. \tag{32}$$

However, the total energy, $E = K + U$, of an isolated system is constant in time. Hence, the above equation becomes

$$\ddot{I} = -2\,U + c, \tag{33}$$

where $c = 4\,E$ is a constant. Suppose that the system expands radially, in a uniform fashion, by some factor $1 + u(t)$. If R is the outer radius of the star, and M its total mass, then $I \propto M\,R^2$ and $U \propto G\,M/R$. Hence, if $R \to R\,(1 + u)$ at constant mass then $I \to I_0\,(1 + u)^2$ and $U \to U_0/(1 + u)$, where I_0 and U_0 are the unperturbed (*i.e.*, $u = 0$) moment of inertia and potential energy, respectively. It follows from (33) that, in the unperturbed (*i.e.*, $d/dt = 0$) state,

$$0 = -2\,U_0 + c. \tag{34}$$

Hence, (33) gives

$$\frac{d^2}{dt^2}[I_0\,(1 + u)^2] = -2\,U_0/(1 + u) + c. \tag{35}$$

Assuming that $|u| \ll 1$, so that the radial oscillations are of relatively small amplitude, we get

$$2\,I_0\,\frac{d^2u}{dt^2} \simeq -2\,U_0 + c + 2\,U_0\,u = -2\,|U_0|\,u, \tag{36}$$

where use has been made of (34) and the fact that $U_0 < 0$. Hence, the radial oscillation equation takes the form

$$\frac{d^2u}{dt^2} = -\frac{|U_0|}{I_0}\,u. \tag{37}$$

Fitzpatrick, Chapter 2

Exercise 2.5

All the particles have the same charge-to-mass ratio:

$$\frac{q_i}{m_i} = \frac{q}{m}$$

If the system is in a uniform magnetic field $\vec{B}$, the equation of motion for the i^{th} particle is

$$m_i \ddot{\vec{r}}_i = \sum_{j \neq i} \frac{k q_i q_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j) + q_i \dot{\vec{r}}_i \times \vec{B}$$

If we sum over i, the left side is the time derivative of the total momentum

$$\sum_i m_i \ddot{\vec{r}}_i = \frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i \right) = \frac{d}{dt} \vec{P}$$

The resulting equation can be written

$$\frac{d}{dt}\vec{P} = \sum_i \sum_{j \neq i} \frac{k q_i q_j}{|\vec{r}_i - \vec{r}_j|} (\vec{r}_i - \vec{r}_j) + \sum_i q_i \dot{\vec{r}}_i \times \vec{B}$$

$$= \sum_{\substack{i \\ i \neq j}} \sum_j \frac{k q_i q_j}{|\vec{r}_i - \vec{r}_j|} \tfrac{1}{2}\left[\cancel{(\vec{r}_i - \vec{r}_j)} + \cancel{(\vec{r}_j - \vec{r}_i)} \right] + \left(\sum_i q_i \dot{\vec{r}}_i \right) \times \vec{B}$$

$$= \frac{q}{m} \vec{P} \times \vec{B}$$

This equation can be written

$$\frac{d}{dt}\vec{P} = -\Omega\,\hat{B}\times\vec{P},$$

$$\Omega = \frac{q}{m}B$$

It implies that $\vec{P}$ processes around the direction $\hat{B}$ with angular frequency Ω.

The total angular momentum parallel to $\vec{B}$ is

$$L_{\parallel} = \vec{L}\cdot\hat{B}$$

$$= \left(\sum_i m_i \vec{r}_i\times\dot{\vec{r}}_i\right)\cdot\hat{B}$$

The moment of inertia about an axis through the origin parallel to $\hat{B}$ is

$$I_{\parallel} = \sum_i m_i(\vec{r}_i\times\hat{B})^2$$

The time derivative of $L_{\parallel}$ is

$$\frac{d}{dt}L_{\parallel} = \left[\sum_i m_i\left(\cancel{\dot{\vec{r}}_i\times\dot{\vec{r}}_i} + \vec{r}_i\times\ddot{\vec{r}}_i\right)\right]\cdot\hat{B}$$

$$= \left(\sum_i \vec{r}_i\times(m_i\ddot{\vec{r}}_i)\right)\cdot\hat{B}$$

Inserting the equations of motion, the becomes

$$\frac{d}{dt}L_{\parallel} = \left(\sum_i \vec{r}_i \times (q_i \dot{\vec{r}}_i \times \vec{B})\right) \cdot \hat{B}$$

$$= -\sum_i q_i (\dot{\vec{r}}_i \times \vec{B}) \times \vec{r}_i \cdot \hat{B}$$

$$= -\frac{q}{m} B \sum_i m_i (\dot{\vec{r}}_i \times \hat{B}) \cdot (\vec{r}_i \times \hat{B})$$

The time derivative of $I_{\parallel}$ is

$$\frac{d}{dt} I_{\parallel} = \sum_i m_i \, 2 (\vec{r}_i \times \hat{B}) \cdot (\dot{\vec{r}}_i \times \hat{B})$$

Thus the time derivative of $L_{\parallel} + \frac{1}{2}\Omega I_{\parallel}$ is

$$\frac{d}{dt}\left(L_{\parallel} + \frac{1}{2}\Omega I_{\parallel}\right)$$

$$= -\frac{q}{m} B \sum_i m_i (\dot{\vec{r}}_i \times \hat{B}) \cdot (\vec{r}_i \times \hat{B})$$

$$+ \frac{1}{2}\left(\frac{q}{m} B\right) 2 \sum_i m_i (\vec{r}_i \times \hat{B}) \cdot (\dot{\vec{r}}_i \times \hat{B})$$

$$= 0$$

Thus $L_{\parallel} + \frac{1}{2}\Omega I_{\parallel}$ is a constant of the motion.