Physics 336K: Newtonian Dynamics
Homework 9: Solutions

11.1

1. Let x_1 and x_2 be the vertical coordinates of the upper and lower masses, respectively, and let the system be in equilibrium when $x_1 = x_2 = 0$. The kinetic energy of the system then takes the form

$$K = \frac{1}{2}\,m\,\dot{x}_1^{\,2} + \frac{1}{2}\,m\,\dot{x}_2^{\,2}. \tag{1}$$

The potential energy (neglecting gravity) of the system is written

$$U = \frac{1}{2}\left[k\,x_1^{\,2} + k\,(x_2 - x_1)^2\right], \tag{2}$$

or

$$U = -\frac{1}{2}\left[-2\,k\,x_1^{\,2} + 2\,k\,x_1\,x_2 - k\,x_2^{\,2}\right]. \tag{3}$$

The mass matrix is

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \tag{4}$$

whereas the force matrix takes the form

$$\mathbf{G} = \begin{pmatrix} -2\,k & k \\ k & -k \end{pmatrix}. \tag{5}$$

We need to solve

$$(\mathbf{G} - \lambda\,\mathbf{M})\,\mathbf{x} = \mathbf{0}, \tag{6}$$

or

$$\begin{pmatrix} -2\,k - \lambda\,m & k \\ k & -k - \lambda\,m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7}$$

The eigenvalue, λ, is determined by setting the determinant of the matrix to zero. This gives

$$(2\,\omega_0^{\,2} + \lambda)\,(\omega_0^{\,2} + \lambda) - \omega_0^{\,4} = 0, \tag{8}$$

where $\omega_0 = \sqrt{k/m}$, which reduces to

$$\lambda^2 + 3\,\omega_0^{\,2}\,\lambda + \omega_0^{\,4} = 0. \tag{9}$$

The solutions are

$$\lambda_\pm = \left(\frac{-3 \mp \sqrt{5}}{2}\right)\omega_0^{\,2}. \tag{10}$$

Hence, the normal frequencies are

$$\omega_\pm = \sqrt{-\lambda_\pm} = \frac{(3 \pm \sqrt{5})^{1/2}\,\omega_0}{\sqrt{2}}. \tag{11}$$

From (7),

$$\frac{x_2}{x_1} = \left(2 + \frac{\lambda}{\omega_0^{\,2}}\right). \tag{12}$$

Thus, for the lower frequency normal mode,

$$\frac{x_2}{x_1} = \left(\frac{\sqrt{5}+1}{2}\right), \tag{13}$$

whereas for the higher frequency mode,

$$\frac{x_2}{x_1} = -\left(\frac{\sqrt{5}-1}{2}\right). \tag{14}$$

Thus, the lower frequency mode is an even mode in which the two masses move in the same direction, whereas the higher frequency mode is an odd mode in which the two masses move in opposite directions.

2. Let θ_1 and θ_2 be the angles of inclination of the upper and lower pendulums, respectively, to the downward vertical. The equilibrium corresponds to $\theta_1 = \theta_2 = 0$. Thus, taking the suspension point as the origin, the horizontal and vertical coordinates of the upper bob are

$$x_1 = l\,\sin\theta_1, \tag{15}$$

$$y_1 = -l\,\cos\theta_1, \tag{16}$$

11.5

5. Let θ and ϕ be the angles subtended by the string and the rod, respectively, with the downward vertical. The equilibrium corresponds to $\theta = \phi = 0$. With the suspension point as the origin, the horizontal and vertical Cartesian coordinates of the center of mass of the rod are

$$x = l\,\sin\theta + (a/2)\,\sin\phi, \tag{67}$$

$$y = -l\,\cos\theta - (a/2)\,\cos\phi, \tag{68}$$

respectively. The kinetic energy of the system is thus

$$K = \frac{1}{2}\,I\,\dot{\phi}^{\,2} + \frac{1}{2}\,m\,(\dot{x}^{\,2} + \dot{y}^{\,2}), \tag{69}$$

where $I = (1/12)\,m\,a^2$ is the moment of inertia of the rod about its center of mass. Hence,

$$K = \frac{1}{2}\left[\frac{1}{3}\,m\,a^2\,\dot{\phi}^{\,2} + m\,l\,a\,\cos(\theta-\phi)\,\dot{\phi}\,\dot{\theta} + m\,l^2\,\dot{\theta}^{\,2}\right]. \tag{70}$$

The potential energy is

$$U = m\,g\,y = -m\,g\,\left[(a/2)\,\cos\phi + l\,\cos\theta\right]. \tag{71}$$

In the limit that ϕ and θ are small angles, we find that

$$K = \frac{1}{2}\left[\frac{1}{3}\,m\,a^2\,\dot{\phi}^{\,2} + m\,l\,a\,\dot{\phi}\,\dot{\theta} + m\,l^2\,\dot{\theta}^{\,2}\right], \tag{72}$$

and

$$U = -\frac{1}{2}\,m\,g\left[-(a/2)\,\phi^2 - l\,\theta^2\right], \tag{73}$$

where we have neglected an unimportant constant term in the expression for U. The mass matrix is

$$\mathbf{M} = \begin{pmatrix} (1/3)\,m\,a^2 & (1/2)\,m\,l\,a \\ (1/2)\,m\,l\,a & m\,l^2 \end{pmatrix}, \tag{74}$$

whereas the force matrix takes the form

$$\mathbf{G} = \begin{pmatrix} -(1/2)\,m\,g\,a & 0 \\ 0 & -m\,g\,l \end{pmatrix}. \tag{75}$$

Thus, the eigenvalue equation reduces to

$$\begin{pmatrix} -(1/2)\,m\,g\,a-(1/3)\,\lambda\,m\,a^2 & -(1/2)\,\lambda\,m\,l\,a \\ -(1/2)\,\lambda\,m\,l\,a & -m\,g\,l-\lambda\,m\,l^2 \end{pmatrix}\begin{pmatrix}\phi\\ \theta\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{76}$$

The eigenvalue, λ, is determined by setting the determinant of the matrix to zero. This gives

$$\left(\frac{1}{2}\,\alpha\,\omega_0^{\,2}+\frac{1}{3}\,\lambda\right)(\omega_0^{\,2}+\lambda)-\frac{1}{4}\,\lambda^2=0, \tag{77}$$

where $\alpha=l/a$ and $\omega_0=\sqrt{g/l}$, which reduces to

$$\lambda^2+2\,(2+3\,\alpha)\,\omega_0^{\,2}\,\lambda+6\,\alpha\,\omega_0^{\,4}=0. \tag{78}$$

The roots are

$$\lambda_\pm=-2\left[\left(1+\frac{3}{2}\,\alpha\right)\pm\left(1+\frac{3}{2}\,\alpha+\frac{9}{4}\,\alpha^2\right)^{1/2}\right]\omega_0^{\,2}. \tag{79}$$

Thus, the normal frequencies are

$$\omega_\pm=\sqrt{2}\left[\left(1+\frac{3}{2}\,\alpha\right)\pm\left(1+\frac{3}{2}\,\alpha+\frac{9}{4}\,\alpha^2\right)^{1/2}\right]\omega_0. \tag{80}$$

From Eq. (76),

$$\frac{\theta}{\phi}=-\left(\frac{\omega_0^{\,2}}{\lambda}+\frac{2}{3\,\alpha}\right), \tag{81}$$

or

$$\frac{\theta}{\phi}=-\frac{1}{3\,\alpha}\left[1-\frac{3}{2}\,\alpha\pm\left(1+\frac{3}{2}\,\alpha+\frac{9}{4}\,\alpha^2\right)^{1/2}\right]. \tag{82}$$

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Exercise 11.6

The equilibrium positions of the 3 masses can be chosen as

$$\vec{r}_1^{\,(eq)} = \left(\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$$

$$\vec{r}_2^{\,(eq)} = \left(0, \frac{a}{\sqrt{3}}\right)$$

$$\vec{r}_3^{\,(eq)} = \left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$$

The general position vectors of the 3 masses are

$$\vec{r}_1 = \left(\frac{a}{2} + x_1, -\frac{a}{2\sqrt{3}} + y_1\right)$$

$$\vec{r}_2 = \left(x_2, \frac{a}{\sqrt{3}} + y_2\right)$$

$$\vec{r}_3 = \left(-\frac{a}{2} + x_3, -\frac{a}{2\sqrt{3}} + y_3\right)$$

The velocities are

$$\dot{\vec{r}}_1 = (\dot{x}_1, \dot{y}_1)$$

$$\dot{\vec{r}}_2 = (\dot{x}_2, \dot{y}_2)$$

$$\dot{\vec{r}}_3 = (\dot{x}_3, \dot{y}_3)$$

The kinetic energy is

$$K = \tfrac{1}{2}m|\dot{\vec{r}}_1|^2 + \tfrac{1}{2}m|\dot{\vec{r}}_2|^2 + \tfrac{1}{2}m|\dot{\vec{r}}_3|^2$$

$$= \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\right)$$

The separations of the particles are

$$|\vec{r}_1 - \vec{r}_2| = \sqrt{\left(\frac{a}{2} + x_1 - x_2\right)^2 + \left(-\frac{a}{2\sqrt{3}} + y_1 - \frac{a}{\sqrt{3}} - y_2\right)^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + 2\frac{a}{2}(x_1 - x_2) + \left(-\frac{\sqrt{3}a}{2}\right)^2 + 2\left(-\frac{\sqrt{3}a}{2}\right)(y_1 - y_2) + \cdots}$$

$$= \sqrt{a^2 + a(x_1 - x_2) - \sqrt{3}a(y_1 - y_2) + \cdots}$$

$$= a + \tfrac{1}{2}(x_1 - x_2) - \frac{\sqrt{3}}{2}(y_1 - y_2) + \cdots$$

$$|\vec{r}_2 - \vec{r}_3| = \sqrt{\left(x_2 + \frac{a}{2} - x_3\right)^2 + \left(\frac{a}{\sqrt{3}} + y_2 + \frac{a}{2\sqrt{3}} - y_3\right)^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + 2\frac{a}{2}(x_2 - x_3) + \left(\frac{\sqrt{3}a}{2}\right)^2 + 2\frac{\sqrt{3}a}{2}(y_2 - y_3) + \cdots}$$

$$= \sqrt{a^2 + a(x_2 - x_3) + \sqrt{3}a(y_2 - y_3) + \cdots}$$

$$= a + \tfrac{1}{2}(x_2 - x_3) + \frac{\sqrt{3}}{2}(y_2 - y_3) + \cdots$$

$$|\vec{r}_3 - \vec{r}_1| = \sqrt{(-a + x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{a^2 - 2a(x_3 - x_1) + \cdots}$$

$$= a - (x_3 - x_1) + \cdots$$

The potential energy expanded to second order in the coordinate $x_1, y_1, x_2, y_2, x_3, y_3$ is

$$U = \tfrac{1}{2}k\left(|\vec{r}_1-\vec{r}_2| - a\right)^2 + \tfrac{1}{2}k\left(|\vec{r}_2-\vec{r}_3| - a\right)^2 + \tfrac{1}{2}k\left(|\vec{r}_3-\vec{r}_1| - a\right)^2$$

$$= \tfrac{1}{2}k\left(\tfrac{1}{2}(x_1-x_2) - \tfrac{\sqrt{3}}{2}(y_1-y_2)\right)^2 + \tfrac{1}{2}k\left(\tfrac{1}{2}(x_2-x_3) + \tfrac{\sqrt{3}}{2}(y_2-y_3)\right)^2$$

$$+ \tfrac{1}{2}k\left(-(x_3-x_1)\right)^2$$

$$= \tfrac{1}{8}k\Big[(x_1-x_2)^2 - 2\sqrt{3}(x_1-x_2)(y_1-y_2) + 3(y_1-y_2)^2$$

$$+ (x_2-x_3)^2 + 2\sqrt{3}(x_2-x_3)(y_2-y_3) + 3(y_2-y_3)^2$$

$$+ 4(x_3-x_1)^2\Big]$$

We can exclude translational modes by choosing the center-of-mass to be at the origin: $\vec{r}_{CM} = (0,0)$
The center-of-mass position vector is

$$\vec{r}_{CM} = \tfrac{1}{3}\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right)$$

$$= \tfrac{1}{3}\left(x_1+x_2+x_3,\ y_1+y_2+y_3\right)$$

The condition that this be equal to $(0,0)$ can be used to eliminate x_2 and y_2 in favor of x_1, x_2, y_1, y_2:

$$x_2 = -x_1 - x_3$$

$$y_2 = -y_1 - y_3$$

The kinetic energy becomes

$$K = \frac{1}{2}m\left[\dot{x}_1^2 + (\dot{x}_1 + \dot{x}_3)^2 + \dot{x}_3^2 + \dot{y}_1^2 + (\dot{y}_1 + \dot{y}_3)^2 + \dot{y}_3^2\right]$$

$$= \frac{1}{2}m\left[2\dot{x}_1^2 + 2\dot{x}_1\dot{x}_3 + 2\dot{x}_3^2 + 2\dot{y}_1^2 + 2\dot{y}_1\dot{y}_3 + 2\dot{y}_3^2\right]$$

The potential energy becomes

$$U = \frac{1}{8}k\Big[(2x_1 + x_3)^2 - 2\sqrt{3}(2x_1 + x_3)(2y_1 + y_3) + 3(2y_1 + y_3)^2$$
$$+ (x_1 + 2x_3)^2 + 2\sqrt{3}(x_1 + 2x_3)(y_1 + 2y_3) + 3(y_1 + 2y_3)^2$$
$$+ 4(x_3 - x_1)^2\Big]$$

$$= \frac{1}{8}k\Big[5x_1^2 + 8x_1x_3 + 5x_3^2 + 15y_1^2 + 24y_1y_3 + 15y_3^2$$
$$+ 2\sqrt{3}(-3x_1y_1 + 3x_3y_3) + 4(x_3^2 - 2x_1x_3 + x_1^2)\Big]$$

$$= \frac{3}{8}k\Big[3x_1^2 + 3x_3^2 + 5y_1^2 + 8y_1y_3 + 5y_3^2$$
$$- 2\sqrt{3}(x_1y_1 - x_3y_3)\Big]$$

We can exclude rotational modes by choosing the rotation angle about the center-of-mass to be 0. We can identify the rotation angle ϕ by expressing the total angular momentum in the form

$$L = I\dot{\phi}$$

The moment of inertia to leading order in x_i, y_i is

$$I = m|\vec{r}_1|^2 + m|\vec{r}_2|^2 + m|\vec{r}_3|^2$$

$$\simeq m|\vec{r}_1^{(eq)}|^2 + m|\vec{r}_2^{(eq)}|^2 + m|\vec{r}_3^{(eq)}|^2$$

$$= m\frac{a^2}{3} \times 3$$

$$= ma^2$$

The total angular momentum to leading order in x_i, y_i is

$$L = m\vec{r}_1 \times \dot{\vec{r}}_1 + m\vec{r}_2 \times \dot{\vec{r}}_2 + m\vec{r}_3 \times \dot{\vec{r}}_3$$

$$\simeq m\left[\frac{a}{2}\dot{y}_1 - \left(-\frac{a}{2\sqrt{3}}\right)\dot{x}_1\right] + m\left[0 \cdot \dot{y}_2 - \frac{a}{\sqrt{3}}\dot{x}_2\right]$$

$$+ m\left[\left(-\frac{a}{2}\right)\dot{y}_3 - \left(-\frac{a}{2\sqrt{3}}\right)\dot{x}_3\right]$$

$$= \frac{ma}{2\sqrt{3}}\left(\dot{x}_1 - 2\dot{x}_2 + \dot{x}_3 + \sqrt{3}\,\dot{y}_1 - \sqrt{3}\,\dot{y}_2\right)$$

Thus the rotation angle is

$$\phi = \frac{1}{2\sqrt{3}\,a}\left(x_1 - 2x_2 + x_3 + \sqrt{3}\,y_1 - \sqrt{3}\,y_2\right)$$

We can use the condition $\phi = 0$ to eliminate $y_1 - y_3$ in favor of x_1, x_2, x_3:

$$y_1 - y_3 = -\frac{1}{\sqrt{3}}\left(x_1 - 2x_2 + x_3\right)$$

$$= \frac{-3}{\sqrt{3}}(x_1 + x_3) + \frac{2}{\sqrt{3}}(x_1 + x_2 + x_3)$$

$$= -\sqrt{3}(x_1 + x_3)$$

We choose the independent coordinates to be x_1, x_3, and $\bar{y} = \frac{1}{2}(y_1 + y_3)$. The appropriate substitution for y_1 and y_2 are

$$y_1 = \bar{y} - \frac{\sqrt{3}}{2}(x_1 + x_3)$$

$$y_3 = \bar{y} + \frac{\sqrt{3}}{2}(x_1 + x_3)$$

The kinetic energy is

$$K = \tfrac{1}{2}m\Big[2\dot{x}_1^2 + 2\dot{x}_1\dot{x}_3 + 2\dot{x}_3^2 + 2\left(\dot{\bar{y}} - \tfrac{\sqrt{3}}{2}(\dot{x}_1 + \dot{x}_3)\right)^2$$

$$+ 2\left(\dot{\bar{y}} - \tfrac{\sqrt{3}}{2}(\dot{x}_1 + \dot{x}_3)\right)\left(\dot{\bar{y}} + \tfrac{\sqrt{3}}{2}(\dot{x}_1 + \dot{x}_3)\right) + 2\left(\dot{\bar{y}} + \tfrac{\sqrt{3}}{2}(\dot{x}_1 + \dot{x}_3)\right)^2\Big]$$

$$= \tfrac{1}{4}m\left[7\dot{x}_1^2 + 10\dot{x}_1\dot{x}_3 + 7\dot{x}_3^2 + 12\dot{\bar{y}}^2\right]$$

The potential energy is

$$U = \frac{3}{8}k\Big[3x_1^2 + 3x_3^2 + 5\left(\bar{y} - \tfrac{\sqrt{3}}{2}(x_1+x_3)\right)^2$$

$$+ 8\left(\bar{y} - \tfrac{\sqrt{3}}{2}(x_1+x_3)\right)\left(\bar{y} + \tfrac{\sqrt{3}}{2}(x_1+x_3)\right) + 5\left(\bar{y} + \tfrac{\sqrt{3}}{2}(x_1+x_3)\right)^2$$

$$- 2\sqrt{3}\,x_1\left(\bar{y} - \tfrac{\sqrt{3}}{2}(x_1+x_3)\right) + 2\sqrt{3}\,x_3\left(\bar{y} + \tfrac{\sqrt{3}}{2}(x_1+x_3)\right)\Big]$$

$$= \frac{3}{8}k\Big[3(x_1^2+x_3^2) + \tfrac{3}{2}(x_1+x_3)^2 + 18\bar{y}^2$$

$$- 2\sqrt{3}(x_1-x_3)\bar{y} + 3(x_1+x_3)^2\Big]$$

$$= \frac{3}{16}k\left[15x_1^2 + 18x_1x_3 + 15x_3^2 - 4\sqrt{3}(x_1-x_3)\bar{y} + 36\bar{y}^2\right]$$

We arrange the 3 remaining coordinate into a column vector:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \bar{y} \end{pmatrix}$$

The kinetic and potential energies can be written

$$K = \tfrac{1}{2}X^T M X$$

$$U = \tfrac{1}{2}X^T \mathbb{K} X$$

where the 3×3 matrices are

$$\mathbb{M} = \tfrac{1}{2}m\begin{pmatrix} 7 & 5 & 0 \\ 5 & 7 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$$\mathbb{K} = \tfrac{3}{8}k\begin{pmatrix} 15 & 9 & -4\sqrt{3} \\ 9 & 15 & 4\sqrt{3} \\ -4\sqrt{3} & 4\sqrt{3} & 36 \end{pmatrix}$$

The equations of motion are

$$\mathbb{M}\ddot{X} = -\mathbb{K}X$$

The eigenvalue λ of $\mathbb{K}$ relative to $\mathbb{M}$ are defined by

$$\mathbb{K}X = \lambda\mathbb{M}X$$

The eigenvalues and eigenvectors can be calculated using Mathemateca. The eigenvalue are $3k/m$ $\frac{3}{2}k/m$, and $\frac{3}{2}k/m$. The corresponding normalized eigenvectors are

$$\begin{pmatrix} -0.655 \\ 0.655 \\ 0.378 \end{pmatrix}, \begin{pmatrix} 0.951 \\ 0.229 \\ 0.209 \end{pmatrix}, \begin{pmatrix} -0.520 \\ 0.775 \\ -0.371 \end{pmatrix}$$

Any orthogonal linear combinations of the last two eigenvectors are also eigenvectors with eigenvalue $\frac{3}{2}k/m$.