Preservice education students' knowledge of division: In need of buttressing

Gordon J. Aubrecht, II
Department of Physics, Ohio State University, Marion, OH 43302-5695
and Columbus, OH 43210-1106, aubrecht@mps.ohio-state.edu

Many of my preservice elementary education students in my Physics by Inquiry classes (for whom Physics by Inquiry material was intended) fall back on the simplest operations without internalizing their understanding of division. I report here on a simple way to supplement the text that forces students to come to grips with the actual meaning of division in terms of whole and package by building more explicitly on the techniques students have already developed to define area and volume operationally.

Arons [A. Arons, "Cultivating the capacity for formal reasoning: Objectives and procedures in an introductory physical science course," *Am. J. Phys.* 44, 834-838 (1976).] pointed out the incomprehension science students exhibit of the basic mathematical operations multiplication and division, and the need to address the problem in physics classes. McDermott et al.'s Physics by Inquiry program [L. C. McDermott et al., *Physics by Inquiry* (New York: Wiley, 1995) (Vol. I: Properties of matter; Heat and temperature; Light and color; Magnets; and Astronomy by sight: the sun, moon, and stars and Vol. II: Electric circuits, Electromagnets, Light and optics, Kinematics; and Astronomy by sight: the Earth and the solar system).] does address this need directly and in detail (by defining two meanings for division in Sec. 9 of Vol. I, one of which is called whole and package reasoning).

Physics by inquiry is based on two pillars: *Elicit-Confront-Resolve*, a theoretical approach to elimination of misconceptions; and hands-on exposure to physical phenomena. It is assumed that many ideas students bring to introductory physics are shared, hard-fixed, misconceptions. This may not always be the case, as many others have argued, but it has provided the guiding paradigm for physics by inquiry. In the elicit-confront-resolve method, students are asked a question that research shows has often been answered incorrectly (speaking scientifically). This elicits the misunderstanding. Then students do an experiment whose result contradicts the specific incorrect answer (this is hands-on learning). They are asked questions that force them to reflect on their understanding and see what was right and wrong about their initial suppositions (the confront phase). Finally, with the help of student statements from interviews quoted in the text, and the instructors' questioning, students resolve the seeming contradictions in a way that builds scientific understanding into their worldviews.

McDermott's research program is based on the pioneering work of Arons, a proponent of confronting student misunderstandings of both mathematics and science. Arons pointed out the reliance of physical models on mathematical language for clarity. Mathematically-inclined students who try to express scientific thought often rely on mathematical relationships, formulas, which may be totally misunderstood, to the exclusion of actual knowledge of how the world works. The less mathematically inclined have no basis except memorization for their understanding of science, which scientists find quite unsatisfactory.

The incomprehension these math-averse students exhibit of the basic mathematical operations multiplication and division needs to be addressed in physics classes.[A. Arons, *Teaching Introductory Physics* (New York: Wiley, 1996), Ch. 1.] Arons has especially pointed to ratio reasoning and fractions as needed foundations for reasoning.

The physics by inquiry program was designed for preservice and inservice teachers, and extensively tested with inservice teachers during summer programs at the University of Washington. Such students' lack of understanding of mathematical thinking would continue to span generations, making the current generations of schoolchildren as afraid of mathematics as their teachers. Fear of mathematics among elementary school teachers is surely responsible for a least a portion of the inability of many elementary students to learn the fundamentals of mathematics and mathematical reasoning.

Student preparation in Properties of matter

In the first module of the physics by inquiry program, issues of density, sinking and floating, and solutions are addressed. Students are first forced to define mass operationally. By an operational definition, we mean that we define a concept by explaining how to measure it. While operational definitions may be expressed in simple language, they set out a process, a series of steps that give directions for measuring the thing, and a criterion by which the measured thing is defined.

Students in the class have already faced the problem of how to define area and volume, in addition to mass, operationally. For example, they end up defining area as follows:

(1) Choose squares whose sides are 1 cm long as standard squares.
(2) Fit the standard squares inside the boundaries of the figure.
(3) Count the number of squares that fit inside the boundaries, estimating the contribution of the partial squares.
(4) The number is the area of the object.

They realize that the standard square could be 1 meter by 1 meter (or anything they decide), but they see in any case how to *tile the area* by using the standard squares. At this point, they are asked to explain why the area of a rectangle is length times width. The aim is to have the students recognize that if a rectangular area is tiled with standard squares, three rows of four columns, for example, then repeated addition—3 + 3 + 3 + 3 + 3, or 4 + 4 + 4—is the same as multiplication.

In a later section, students are led to define volume analogously.

Some students, despite having taken many mathematics courses in their careers, only realize what multiplication is in my class when confronted with a need to produce an explanation such as this.

The students taking my physics by inquiry class are mostly elementary education majors (or are aiming ultimately to teach in elementary school). They are not sure of anything having to do with mathematics. Many of them have struggled with mathematics for years—and lost!

While the physics by inquiry program at the University of Washington has succeeded in getting students there to an understanding of fractions and proportional reasoning with this background, I found myself dissatisfied with my own students' understanding even after instruction using the physics by inquiry materials. I describe here a way I found to make sure that the ideas implicit in the text are made explicit.

Whole and package reasoning problem

Consider the following problem encountered in thinking about density (but before the word density has been mentioned):

A piece of metal has a mass of 125 g and a volume of 32 cm^3. Draw a diagram that shows the thinking involved.

A. What is the mass of 1 cm^3 of this metal?
B. What is the mass of 12.3 cm^3 of this same type of metal?
C. What is the volume of 80 g of this metal?
D. What is the mass of 134.2 cm^3 of this same type of metal?
E. What is the volume of 225 g of this metal?

How would *you* work the problem? There is a natural tendency to do part (A) and find density, then use the density to solve each remaining part.

There is nothing wrong with the method of using density—*if* you're sure of what you're doing. As I have already pointed out, my students were not sure of mathematical meaning.

The idea of the problem is to get students thinking of the meaning of the division in the definition of density, and of the meaning of fractions in general. McDermott (following Arons) defines two meanings of division (she applies it to this purpose), The first is the standard that is done by mathematics students everywhere, probably with no idea of why. In the second definition, which comes after an extended (2 page) discussion of how to think in terms of packages, we find the idea of the package.

(1) "5 cm^3 has a mass of 15 g. We can find the mass of just 1 cm3 by dividing 15/5."

(2) "We want to find out how many packages of 3 g fit into 60 g. We can do this by dividing 60/3."

Note that both the physical quantities in the second definition have the same unit, while they are different in the first. The discussion includes instructions for drawing the situation represents the volume as an area and shows the packages tiling the space just as was done for the operational definition of area.

The book also writes the one meaning for multiplication in terms of packages: "We need to add up 120 of these 3 g packages. A quick way to do this is to multiply 120 x 3."

Because area had been assimilated by the students, and volume had been discussed in the preceding section as very different from area, there is only a small chance that students would be confused by the representation of a volume by an area. Discussions with students during the checkpoints verify that area-volume confusion is not a cause for their original difficulties with whole and package reasoning, as defined by McDermott.

The implicit message of the package idea is that in division, we are tiling the space in pieces, and counting the number of tilings, just as in the previous finding of the area by our operational definition.

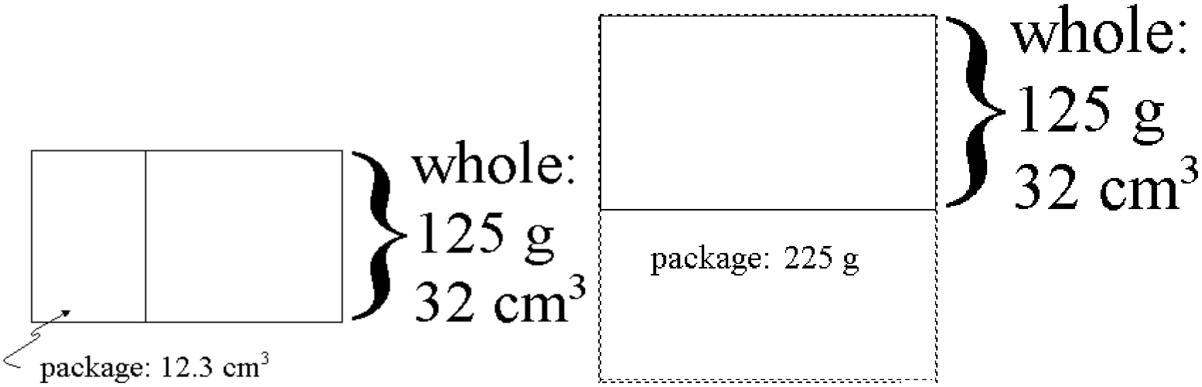
The students used the method of definition (1) exclusively, even in the exercises in which they were supposed to apply the definition (2). I noted this with concern the first time I taught the course (with a colleague), and we asked extra questions and tried to use the text exercises to get students to understand the whole and package idea, but I felt dissatisfied with the outcome. The implicit analogy between division and multiplication suggested by the picture was, however, not grasped by students.

In subsequent quarters, I attempted to force students to go back and recalculate the parts of the questions in terms of whole and package reasoning. I remained dissatisfied, convinced that they did not “get” it. For my students, those two book pages did not seem to be enough, though obviously they were sufficient for the University of Washington inservice teachers on whom the materials were tested.
At this point students have not yet defined density, the idea they prefer to use to calculate, because of their prior assumed lack of understanding of the meaning of division, one meant to be remedied by the book’s presentation. In the following section, they are asked to explain what density means in words.

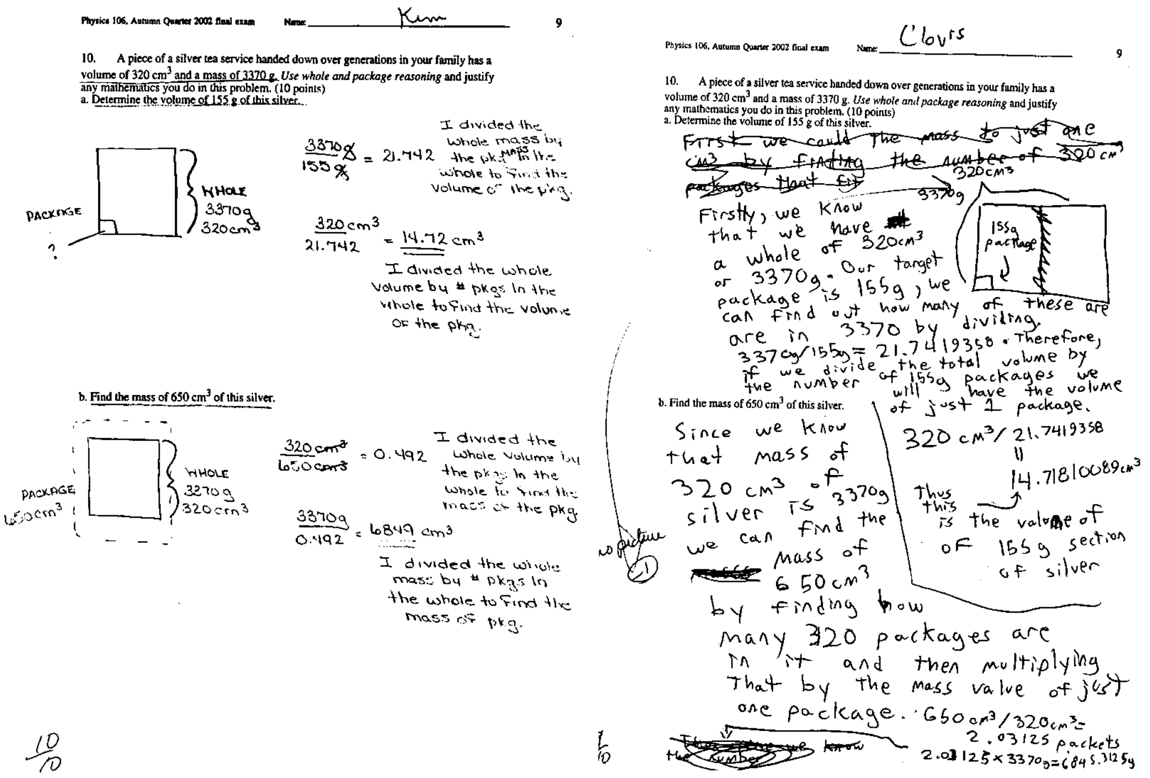
We expect them to say something like “for a homogeneous material, every 1 cm^3 piece contains the same amount of mass. Density measures this ratio of mass to volume.”

I allow students to do the exercises in Sec. 9 as they wish (using whatever method they wish—almost never whole and package reasoning).

Eventually, I made a procedure out of the material on those two pages, extending McDermott’s discussion by having them do an extra worksheet, making the students draw diagrams (see the picture we would expect the student to draw for parts B and E below) and then gave them practice in showing how the method would work under different circumstances. The extra time spent by students to do this material is no more than 2 hours.



Two whole and package reasoning diagrams in formulating an answer.



Physics 106, Autumn Quarter 2002 final exam Name: Kim 9

10. A piece of a silver tea service handed down over generations in your family has a volume of 320 cm^3 and a mass of 3370 g. *Use whole and package reasoning* and justify any mathematics you do in this problem. (10 points)

a. Determine the volume of 155 g of this silver.

$\frac{3370 \cancel{g}}{155 \cancel{g}} = 21.742$ I divided the whole mass by the pkg mass in the whole to find the volume of the pkg.

$\frac{320 cm^3}{21.742} = 14.72 cm^3$

I divided the whole volume by # pkgs in the whole to find the volume of the pkg.

b. Find the mass of 650 cm^3 of this silver.

$\frac{320 \cancel{cm^3}}{650 \cancel{cm^3}} = 0.492$ I divided the whole volume by the pkgs in the whole to find the mass of the pkg

$\frac{3370g}{0.492} = 6849 cm^3$

I divided the whole mass by # pkgs in the whole to find the mass of pkg.

10/10

Physics 106, Autumn Quarter 2002 final exam Name: Clovis 9

10. A piece of a silver tea service handed down over generations in your family has a volume of 320 cm^3 and a mass of 3370 g. *Use whole and package reasoning* and justify any mathematics you do in this problem. (10 points)

a. Determine the volume of 155 g of this silver.

~~First we could the mass to just one cm^3 by finding the number of 320 cm^3 packages that fit~~

Firstly, we know that we have a whole of 320cm^3 or 3370g. Our target package is 155g, we can find out how many of these are in 3370 by dividing. 3370g/155g = 21.7419358. Therefore, if we divide the total volume by the number of 155g packages we will have the volume of just 1 package.

320 cm^3/ 21.7419358 = 14.71810089cm^3

Thus this is the volume of 155g section of silver

b. Find the mass of 650 cm^3 of this silver.

Since we know that mass of 320 cm^3 of silver is 3370g we can find the mass of 650cm^3 by finding how many 320 packages are in it and then multiplying that by the mass value of just one package. 650cm^3/320cm^3 = 2.03125 packets

2.03125 × 3370g = 6845.3125g

~~Thus~~ ~~know~~ ~~the number~~

no picture -1

1/10

Actual samples of a final examination question involving whole and package reasoning for two "typical" students: Kim and Clovis.

Let's see how two typical students did a similar whole and package reasoning problem (preceding pages). One of the students was an A student, the other a C student. Note that both got respectable scores on this problem. You cannot tell which one, however, got an A in the course from the sample shown.

Also (anecdotally), the preferred description for density in Sec. 10 seems to come reasonably effortlessly to the groups after they have been through the extra exercise, while in the "old days" it seemed to take forever.

Students after doing the extra worksheet are at least able to follow the steps, no matter whether they are “good” or less “good” students. They know when to do what in the calculations, and examples of student work on exams and class discussions with them confirm that they understand division and fractions better after this extended treatment than before. They often use the method even when not requested to, and many students say this is the first time in their lives they have ever understood division.

Additional handout used with Sec. 9 of Module 1, Vol. I of *Physics by Inquiry.*

Exercise 9.4A

Explain every step in the following problem **using whole and package reasoning**. (The whole is the same for each part.) A piece of clay has a mass of 14.1 g and a volume of 12.2 cm^3.

A. What is the mass of 1 cm^3 of this clay? Draw a diagram that shows the thinking involved.
B. What is the mass of 15.2 cm^3 of this same type of clay? Draw a diagram that shows the thinking involved.
C. Suppose we have a 68.8 g piece of the same clay. By how much would the mass increase if we added a lump of clay with a volume of 3 cm^3?

√

Exercise 9.4B

Explain every step in the following problem using **using whole and package reasoning**. A piece of metal has a mass of 125 g and a volume of 32 cm^3.

A. What is the mass of 1 cm^3 of this metal? Draw a diagram that shows the thinking involved in all parts.
B. What is the mass of 12.3 cm^3 of this same type of metal?
C. What is the volume of 80 g of this metal?
D. What is the mass of 134.2 cm^3 of this same type of metal?
E. What is the volume of 225 g of this metal?

Exercise 9.4C

A piece of wax has a mass of 10.20 g and a volume of 14.20 cm^3. What is the mass of 1.50 cm^3 of the wax? What is the volume of 22.22 g of the wax? Explain.

Exercise 9.4D

Explain how the pictures we use to illustrate whole and package reasoning resemble those used for finding the area operationally. Is this a coincidence? What can you say about using an area to represent a volume or a mass?

√