Getting students to come to grips with the meaning of division

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Abstract:
Many years ago, Arons[1] pointed out the incomprehension science students exhibit of the basic mathematical operations multiplication and division, and the need to address the problem in physics classes. McDermott et al.'s *Physics by Inquiry* program [2] does address this need directly and in detail (by defining two meanings for division in Sec. 9 of Vol. I, one of which is called whole and package reasoning, and is the focus of this talk). However, many students in my classes still fell back on the simplest operations without internalizing their understanding. I report here on ways to supplement the text that force students to come to grips with the actual meaning of division in terms of whole and package.

1. A. Arons, "Cultivating the capacity for formal reasoning: Objectives and procedures in an introductory physical science course," *Am. J. Phys.* **44**, 834-838 (1976). See also A. B. Arons, *Teaching Introductory Physics* (New York: J. Wiley & Sons, 1996), Ch. 1.
2. L. C. McDermott et al., *Physics by Inquiry* (New York: Wiley, 1995) (Vols. I and II).

Consider the following problems encountered in thinking about density (but before the *word* density has been mentioned):

A piece of metal has a mass of 125 g and a volume of 32 cm^3. Draw a diagram that shows the thinking involved.

A. What is the mass of 1 cm^3 of this metal?
B. What is the mass of 12.3 cm^3 of this same type of metal?
C. What is the volume of 80 g of this metal?
D. What is the mass of 134.2 cm^3 of this same type of metal?
E. What is the volume of 225 g of this metal?

Audience participation. You do this problem.

I will wait a short while for you to get your answer.

How did *you* work this problem?

There is a natural tendency to do part (A) and find density, then use the density to solve each remaining part. There is nothing wrong with this method—*if you're sure of what you're doing*.

My students are mostly elementary education majors (or are aiming ultimately to teach in elementary school). Many of them have struggled with mathematics for years—and *lost*!

They are not sure of anything having to do with mathematics.

The idea of the problem is to get students thinking of the *meaning* of the **division** in the definition of density, and of the meaning of *fractions* in general.

McDermott (following Arons) defines two meanings of **division** (she applies it to this purpose),

(1) "5 cm^3 has a mass of 15 g. We can find the mass of just 1 cm^3 by dividing 15/5."

(2) "We want to find out how many packages of 3 g fit into 60 g. We can do this by dividing 60/3."

and one meaning for **multiplication**,

"We need to add up 120 of these 3 g packages. A quick way to do this is to multiply 120 x 3."

What *is* whole and package reasoning?

The questions about the metal above are posed after an extended (2 page) discussion of how to think in terms of packages.

For my students, two pages were not enough.

They did not “get” it.

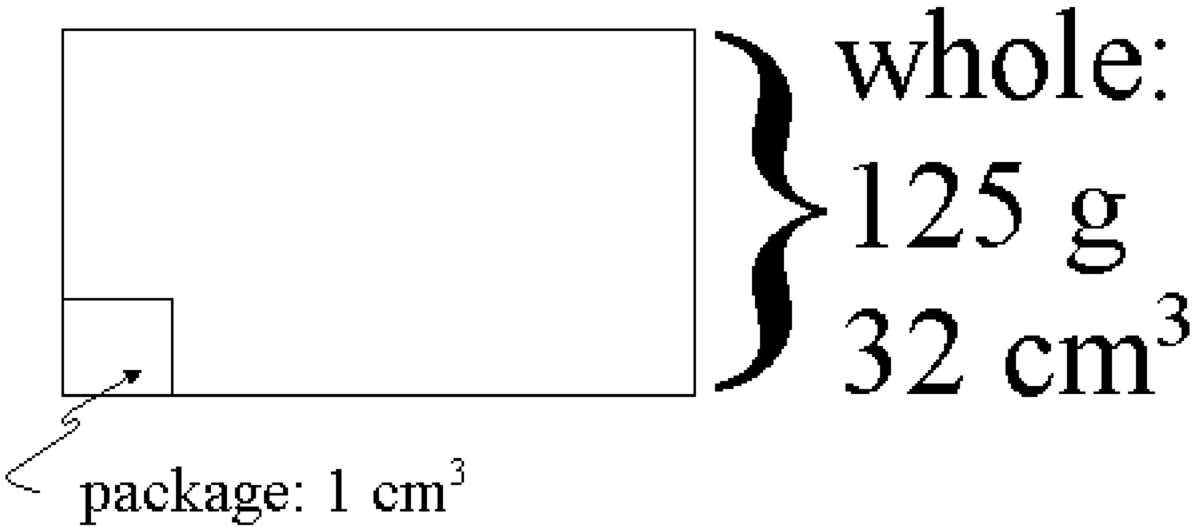
And, of course, at this point they have not defined density because of that.

What to do?

I made a *procedure* out of it, extending McDermott et al.'s discussion, and then gave them practice to show how the method would work under different circumstances.

A. What is the mass of 1 cm^3 of this metal?

First, draw a diagram (for each part).



Now, the number of packages in the whole is the whole volume divided by the volume of the package:

$$32\ cm^3/1\ cm^3 = 32.$$

This number, 32, is a pure number, the number of packages.

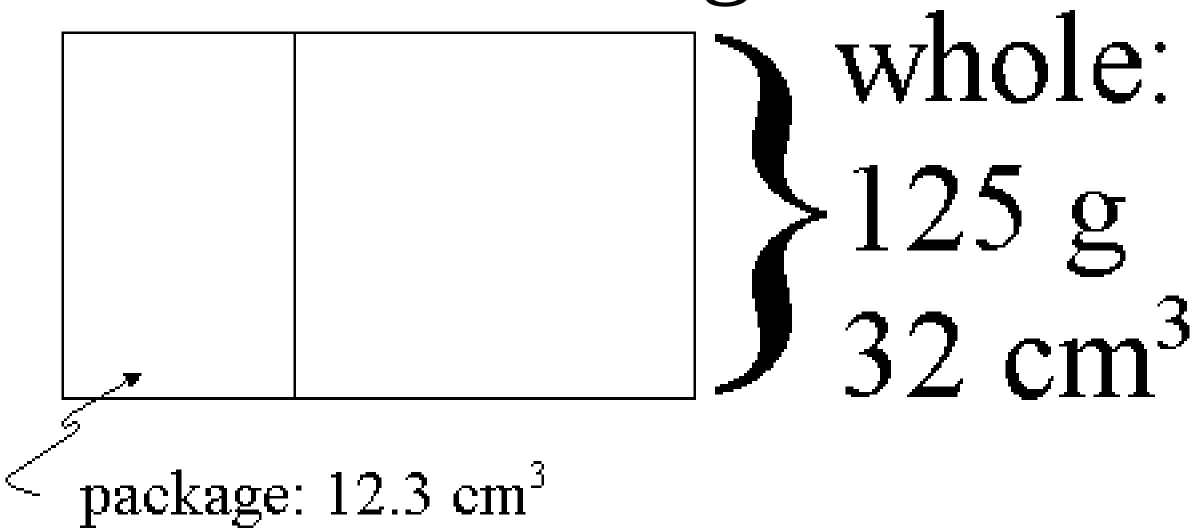
To find the mass in one package, we divide the whole mass by the number of packages:

$$125 \text{ g}/32 = 3.9 \text{ g}.$$

(Of course, this implies the density is 3.9 g/cm^3.)

B. What is the mass of 12.3 cm^3 of this same type of metal?

First, draw a diagram.



The number of packages in the whole is the whole volume divided by the volume of the package:

32 cm^3/12.3 cm^3 = 2.6.

This number, 2.6, is a pure number, the number of packages.

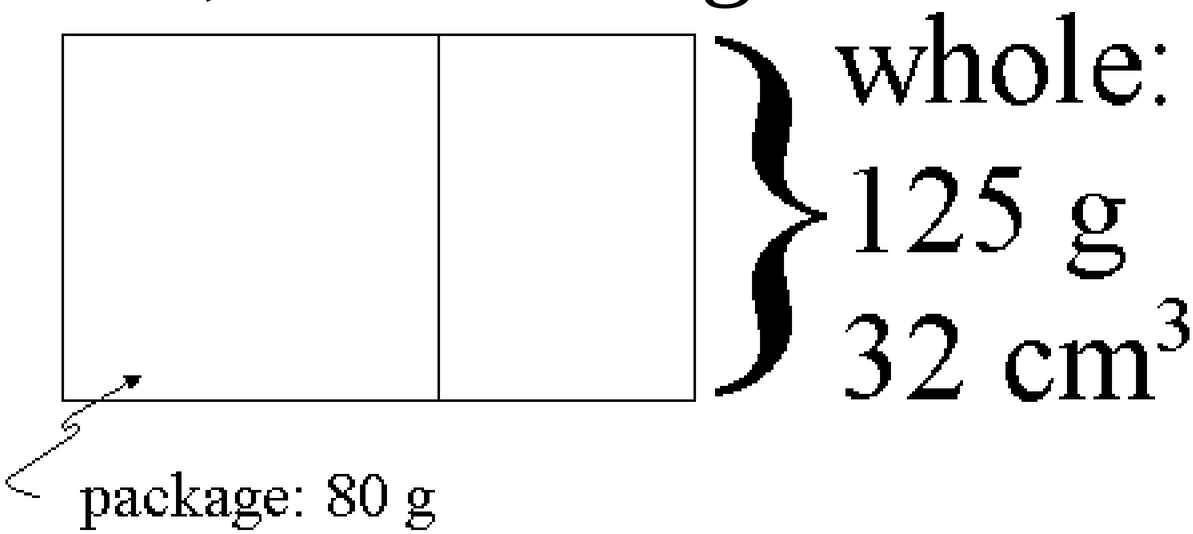
To find the mass in one package, we divide the whole mass by the number of packages:

$$125 \text{ g}/2.6 = 48 \text{ g}.$$

(This does *not* give the density, but before treatment, many students would not realize that.)

C. What is the volume of 80 g of this metal?

First, draw a diagram.



The number of packages in the whole is the whole mass divided by the mass of the package:

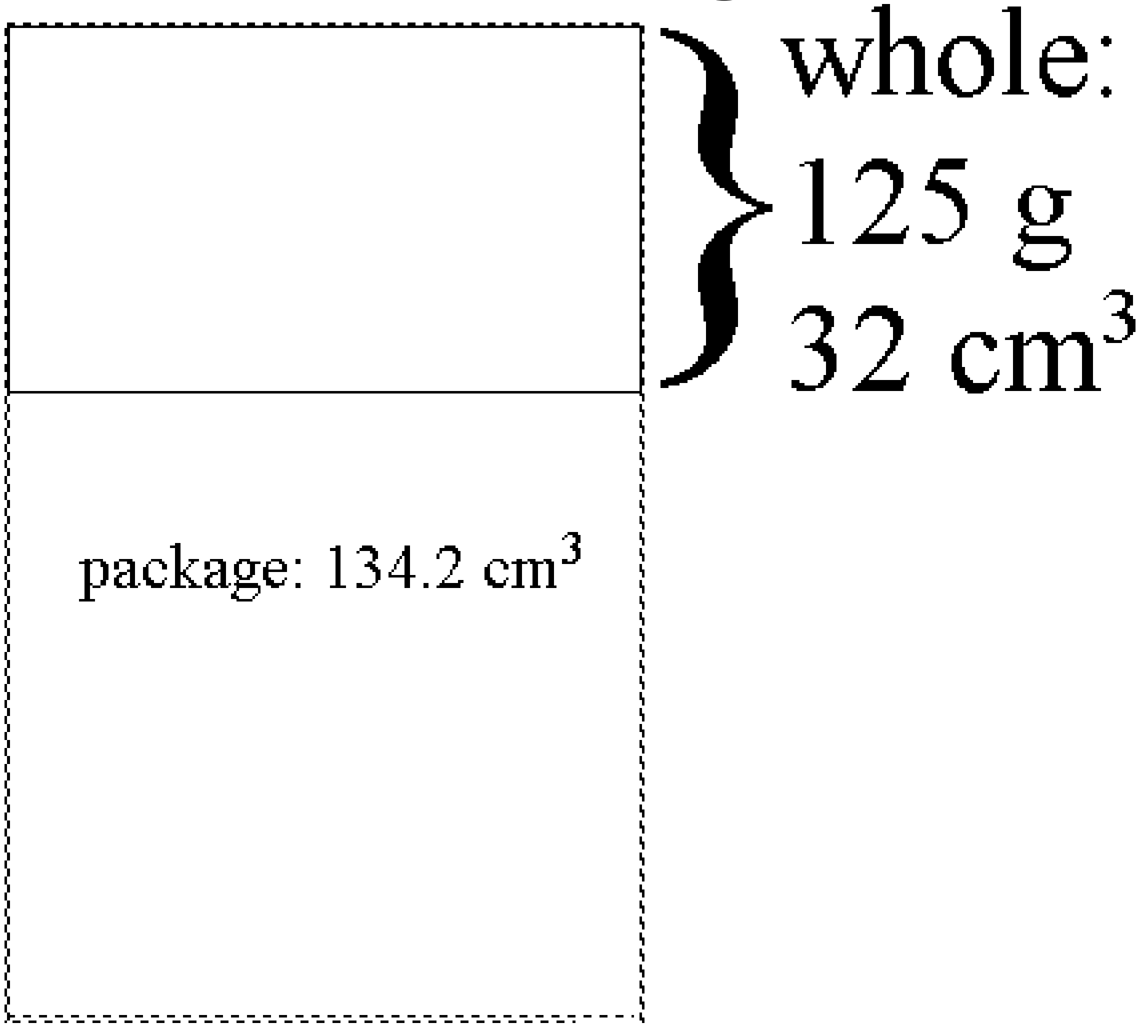
125 g/80 g = 1.56.

This number, 1.56, is a pure number, the number of packages.

To find the volume in one package, we divide the whole volume by the number of packages:
$32\ cm^3/1.56 = 20.5\ cm^3$.

D. What is the mass of 134.2 cm^3 of this same type of metal?

First, draw a diagram.



The number of packages in the whole is the whole volume

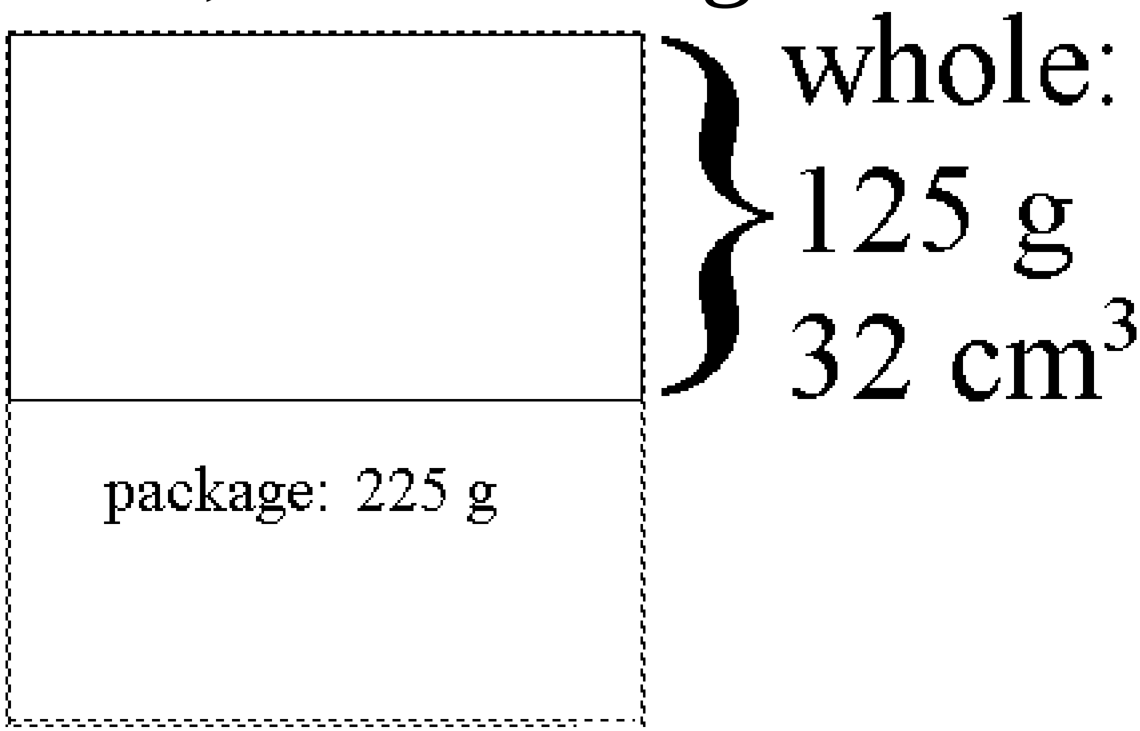
divided by the volume of the package:
$32\ cm^3/134.2\ cm^3$
= 0.238.

This number, 0.238, is a pure number, the number of packages.

To find the mass in one package, we divide the whole mass by the number of packages:
125 g/0.238 = 524 g.

E. What is the volume of 225 g of this metal?

First, draw a diagram.



The number of packages in the whole is the whole mass divided by the mass of the package:

125 g/225 g = 0.56.

This number, 0.56, is a pure number, the number of packages.

To find the volume in one package, we divide the whole volume by the number of packages:
32 cm^3/0.56 = 57.6 cm^3.

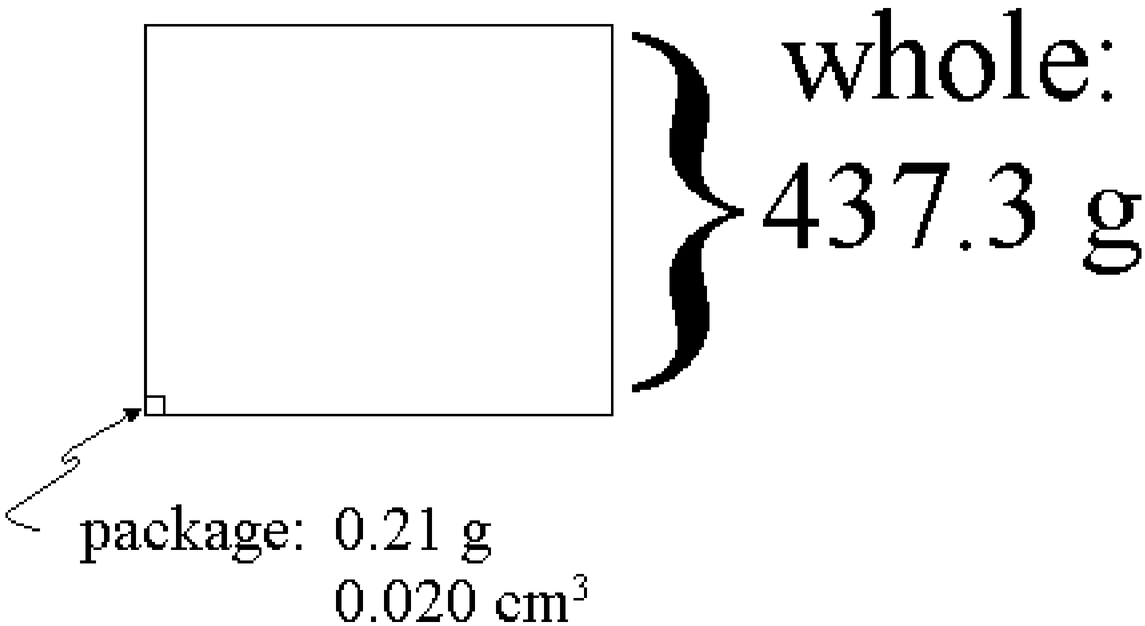
So, though whole and package reasoning might be clumsier in certain cases than the knee-jerk way a student would ordinarily attempt to work a problem, it works *in all cases*, whether the package is smaller or bigger than the whole. We now can easily understand the role of division.

Finally, we give a slightly different example of the use of whole and package reasoning.

Your family has an ornate antique tea service that you suspect is made of silver. Since you wish not to damage the set more than necessary to test, you measure its mass on a very accurate balance to be 437.3 g, and cut out a very small piece from inside one of the legs of the stand. It has a mass of 0.21 g and a volume of 0.02 cm^3. What is the volume of the tea service? Was it silver?

So, what do we do first?

First, draw a diagram!



The number of packages in the whole is the whole mass divided by the mass of the package:
437.3 g/0.21 g = 2082

This pure number, 2082 (2082.4 rounded off), is the number of packages in the whole.

We are trying to find the volume of the whole. Each of the 2082 packages has a volume of 0.020 cm^3. We must *add the volumes* of each of the packages to find the total volume.

The quick way to do this is to multiply

$$2082 \times 0.020\ cm^3 = 41.65\ cm^3.$$

The tea service has a volume of 41.65 cm^3.

To see if it's silver, we need its density. Our *operational definition* (op. def.) of density is

$$\frac{\text{op. def. mass}}{\text{op. def. volume}},$$

so we take

$$\text{density} = \frac{0.21\text{ g}}{0.020\text{ cm}^3}$$

$$= 10.5\text{ g/cm}^3.$$

It *is* silver, because the tables give this as silver's density.

Let’s see how two typical students did a similar whole and package reasoning problem.

Our experience is that the students are at least able to follow the steps and know when to do what in the calculations, and discussions with them confirm that they understand division and fractions better *after this extended treatment*.