PHYSICS 829

Home Work Assignment # 6 5/27/2011

Due: Fri., June 3, 2011 **in class**.

1. *Continuity equation for Dirac equation:* Shankar Ex. 20.1.1 (p. 566)

2. *Dipole Selection Rules:* Show, using the approach suggested below, that the dipole matrix elements $\langle n', \ell', m' | \mathbf{r} | n, \ell, m \rangle$ are non-zero only for

$$\Delta m = 0, \pm 1 \qquad \text{and} \qquad \Delta \ell = \pm 1$$

where $\Delta m = m' - m$ and $\Delta \ell = \ell' - \ell$.

There are many ways to derive this very important result. The most general is to use the Wigner-Eckart theorem (discussed, but not proved, in Shankar Sec. 15.3 – a Section that we did not cover in class). Another way is to use properties of spherical harmonics and just do the integrals involved. Here I lead you through an algebraic approach, which is slightly tedious, but straightforward.

(a1) Using the matrix elements of $[L_z, z]$ show that $\langle n', \ell', m' |\, z \,| n, \ell, m \rangle = 0$ unless $m' = m$.

(a2) Using the commutators $[L_z, x]$ and $[L_z, y]$ find relations between the matrix elements $\langle n', \ell', m' |\, x \,| n, \ell, m \rangle$ and $\langle n', \ell', m' |\, y \,| n, \ell, m \rangle$. Hence show that either $(m' - m)^2 = 1$ or else both $\langle n', \ell', m' |\, x \,| n, \ell, m \rangle = \langle n', \ell', m' |\, y \,| n, \ell, m \rangle = 0$.

(a3) Thus conclude that $\Delta m = 0, \pm 1$.

(b1) Calculate $[L^2, z]$ and using $\mathbf{r} \cdot \mathbf{L} = 0$, find $\left[L^2, [L^2, z]\right]$. Hence deduce that

$$\left[L^2, [L^2, \mathbf{r}]\right] = 2\hbar^2 \left(\mathbf{r} L^2 + L^2 \mathbf{r}\right).$$

(b2) Using matrix elements of this double commutator between the states $|n', \ell', m'\rangle$ and $|n, \ell, m\rangle$ show that the dipole matrix element vanishes unless $\Delta \ell = \pm 1$.

3. *Spontaneous Emission:* Here we solve the problem in a slightly different way from the way we did it in class. Let the Hamiltonian of the system be $H = H_{\text{atom}} + H_{\text{EM}} + H_{\text{int}}$, where the first two terms are the Hamiltonians of the atom and that of the EM fields. The interaction Hamiltonian in the electric dipole approximation is given by

$$H_{\text{int}} = -\boldsymbol{\mu} \cdot \mathbf{E}.$$

Here $\boldsymbol{\mu}$ is the dipole moment operator of the atom and $\mathbf{E}$ the electric field operator at the position of the atom.
Treat H_{int} in lowest order perturbation theory using Fermi's golden rule.

(a) Let us use 'obvious' notation for the unperturbed initial and final states: $|\Psi_i\rangle = |i\rangle \bigotimes |0\rangle$ and $|\Psi_f\rangle = |f\rangle \bigotimes |n_m = 1\rangle$ where $|i\rangle, |f\rangle$ are atomic states, $|0\rangle$ is EM vacuum and $|n_m = 1\rangle$ has one-photon in the mode 'm' with frequency ω, where $\hbar\omega = \epsilon_i - \epsilon_f$. Show that

$$|\langle \Psi_f | H_{\text{int}} | \Psi_i \rangle|^2 = 2\pi\hbar\omega |\langle f | \boldsymbol{\mu} \cdot \mathbf{u}_m | i \rangle|^2$$

where $\mathbf{u}_m$ is the "mode function" (introduced in class).

(b) Show that the density of final states in a box of volume L^3 is given by

$$\rho(\epsilon_f) = \frac{L^3 \omega^2}{\pi^2 \hbar c^3}.$$

(c) Using the Golden Rule show that the rate for spontaneous emission is given by

$$R_s = \frac{4\omega^3}{\hbar c^3} |\langle f | \boldsymbol{\mu} | i \rangle|^2 \frac{L^3}{3} |\mathbf{u}_m|^2.$$

Here the factor 1/3 comes from the angular average of $(\boldsymbol{\mu} \cdot \mathbf{u}_m)^2$.
If the mode function is a plane wave $|\mathbf{u}_m|^2 = 1/L^3$, and we recover the result we have earlier derived using Einstein A and B coefficients in HW #5.

Note that the spontaneous emission rate can be suppressed significantly over its free space value by placing an atom in a cavity in which there are no modes with a frequency satisfying $\hbar\omega = \epsilon_i - \epsilon_f$. In the experiment of Hulet, Hilfer and Kleppner, PRL 55, 2137 (1985), this technique was used to increase the lifetime of an excited state by a factor of 20.