PHYSICS 829

Home Work Assignment # 4 4/25/2011

Due: Mon., May 2, 2011.

1. Shankar Ex. 18.2.4 (p. 478)

2. Rabi oscillations: Consider a two-level system with energies $\hbar\omega_a$ and $\hbar\omega_b$ corresponding to eigenstates $|a\rangle$ and $|b\rangle$ respectively. Let the system be in its ground state $\hbar\omega_a$ at time $t = 0$. The goal is to understand the effect of a *constant* perturbation H' that is turned on for the time interval $0 < t < T$. First we will solve the problem exactly and then we will use use time-dependent perturbation theory to find an approximate solution.

(a) Let the only non-zero matrix elements of H' be $\langle a|H'|b\rangle = \langle b|H'|a\rangle = \hbar W$, and let $\omega_0 = \omega_b - \omega_a > 0$. We can write the state of the system for time $t > 0$ as

$$|\Psi(t)\rangle = c_a(t)e^{-i\omega_a t}|a\rangle + c_b(t)e^{-i\omega_b t}|b\rangle.$$

Find the *exact* equations for $dc_a(t)/dt$ and $dc_b(t)/dt$, written in terms of $c_a(t), c_b(t), W$ and ω_0.

(b) Solve the equations obtained above – without making any approximations – and determine $c_b(t)$ for $0 < t < T$.

(c) Hence find the exact expression for the transition probability $P_{a\to b}(T)$ to go from state $|a\rangle$ at $t = 0$ to $|b\rangle$ at time T. What is the frequency of the resulting *Rabi oscillations*?

(d) Solve the equations derived in (a) using *first order perturbation theory.*

(e) Compare the approximate perturbative answer with the exact result, and discuss when perturbation theory is valid.

3. Spin Resonance: Here we will compare the exact solution of a $S = 1/2$ particle in a static magnetic field $B_0\hat{\mathbf{z}}$, in the presence of a a time-dependent field (solved last quarter) with the approximate solution obtained from time-dependent perturbation theory. The unperturbed Hamiltonian is

$$H_0 = -\frac{1}{2}\hbar\gamma B_0\sigma_z,$$

where γ is the gyromagnetic ratio. The perturbing Hamiltonian

$$H_1(t) = -\frac{1}{2}\hbar\gamma B_1[\sigma_x \cos(\omega t) - \sigma_y \sin(\omega t)]$$

describes a magnetic field of magnitude B_1 rotating in the x-y plane with angular frequency ω. Let the initial state of the system be $|i\rangle = |+\rangle$, corresponding to spin up with energy $E_i = -\hbar\gamma B_0/2$.

Last quarter we showed that the probability for the spin to be flipped into the down state $|f\rangle = |-\rangle$, at the end of an interval $0 \le t \le T$ during which the perturbation acts, is given by

$$|a_f(T)|^2 = \frac{\omega_1^2}{\Omega^2}\sin^2\left(\frac{\Omega T}{2}\right)$$

where

$$\omega_0 = \gamma B_0, \quad \omega_1 = \gamma B_1, \quad \text{and} \quad \Omega = \sqrt{(\omega_0 - \omega)^2 + \omega_1^2}$$

(a) Re-derive the above result using the following steps. Transform to a frame rotating with angular frequency ωt about the z-axis. Show that the effective Hamiltonian in this frame is time-independent, and can be written in the form $H_{\text{eff}} = -\hbar\Omega\vec{\sigma}\cdot\hat{\mathbf{u}}/2$ for a suitable chosen unit vector $\hat{\mathbf{u}}$. Now time evolve the system from $0 \le t \le T$, and determine $|a_f(T)|^2$.

(b) Next, use first order perturbation theory to determine $a_f(T)$.

(c) To compare the perturbative result with the exact answer, first consider the case away from resonance: $\omega_0 \neq \omega$. Show that the perturbative result is then valid for a "sufficiently weak" field B_1. What is the precise condition on B_1?

(d) Next compare the perturbative and exact results on resonance $\omega_0 = \omega$. Show that in this case, no matter how weak B_1 is, perturbation theory will fail after a "sufficiently long" time T. What is the precise condition on T for perturbation theory to be valid?

(e) Finally, expand the perturbative and exact answers to lowest order in T. Show that perturbation theory is always valid for "sufficiently short" times, no matter how strong B_1. Again, state this condition precisely.