PHYSICS 828

Home Work Assignment # 7 3/4/2011

Due: Fri., Mar. 11, 2011.

Completed assignments are due in class on Friday.

1. *Anharmonic oscillator:* Shankar Ex. 17.2.1 (p. 457).

2. Shankar Ex. 17.2.2 (p. 457).

3. Shankar Ex. 17.2.3 (p. 457).
(1) Part (1) is an E & M problem that you do not need to solve for this HW, but you should understand where this result comes from and what it means. For this part, just make a sketch of $V(r)$ as a function of r noting how it deviates from the Coulomb potential.
(2) After doing part (2), make a numerical estimate of the effect using $R \simeq 10^{-15}$m for the "size" of the nucleus.
(3) How does this estimate change for a muonic hydrogen atom in which μ^- muon of mass $\simeq 207m$ orbits a proton?

4. Shankar 17.2.4 (page 457).

5. *Van der Waals force between neutral atoms:* Consider two Hydrogen atoms, which we label A and B, whose nuclei are a distance R apart. Let $\mathbf{r}_A$ and $\mathbf{r}_B$ be the position vectors of the electrons associated with each atom. We are interested in the case where $R \gg |\mathbf{r}_A|, |\mathbf{r}_B|$. Since the two electrons are very well separated, with no overlap in their wavefunctions, we are justified in ignoring the antisymmetry of the two-electron state.
The two widely separated, neutral atoms interact through their dipole moments $e\mathbf{r}_A$ and $e\mathbf{r}_B$. From elementary E & M one can show that the dipole-dipole interaction energy is given by

$$W = \frac{e^2}{R^3}[\mathbf{r}_A \cdot \mathbf{r}_B - 3(\mathbf{r}_A \cdot \hat{\mathbf{n}})(\mathbf{r}_B \cdot \hat{\mathbf{n}})],$$

where $\hat{\mathbf{n}}$ is the unit vector along the line joining A to B.

Thus the Hamiltonian of the system is given by

$$\mathcal{H} = \mathcal{H}_A + \mathcal{H}_B + W$$

where the first two terms correspond to the individual H-atoms. We will treat W as the small perturbation (given the large R) on the exactly solved problem $\mathcal{H}_0 \equiv \mathcal{H}_A + \mathcal{H}_B$. Using obvious notation, we write the eigenstates of $\mathcal{H}_0$ as $|\psi^A_{n,\ell,m};\psi^B_{n',\ell',m'}\rangle$ with corresponding energies $(E_n + E_{n'})$.

(1) You are free to choose $\hat{\mathbf{n}}$ along the z-axis. Show that this leads to

$$W = \frac{e^2}{R^3}(X_A X_B + Y_A Y_B - 2Z_A Z_B)$$

where X_A is the operator corresponding to the x-component of $\mathbf{r}_A$, etc.
(2) Show that the first order correction to the ground state energy vanishes. Can you understand this in physical terms?
(3) Show that the second order correction to the ground state energy is of the form $(-C/R^6)$.
(4) Next we will obtain an order-of-magnitude estimate of the coefficient C of part (3). We make the simple approximation $(2E_1 - E_n - E_{n'}) \simeq 2E_1$ for the denominator in the expression for the second order correction. Show that we can then write

$$C \simeq \frac{e^4}{2|E_1|}\langle\psi^A_{1,0,0};\psi^B_{1,0,0}|(X_A X_B + Y_A Y_B - 2Z_A Z_B)^2|\psi^A_{1,0,0};\psi^B_{1,0,0}\rangle.$$

(5) (a) Exploit the spherical symmetry of the $1s$ state to argue that all 'cross terms' (involving $X_A Y_A$ etc.) in the expression above vanish.
(b) The only non-zero terms are those that involve $\langle\psi^\alpha_{1,0,0}|X^2_\alpha|\psi^\alpha_{1,0,0}\rangle$, where $\alpha = A, B$ and similar terms with X replaced by Y or Z. Show, by symmetry, that each of these is *one-third* the expectation value of $\mathbf{R}^2_\alpha$.
(c) Evaluate the last integral and show that $C = 6e^2 a_0^5$ where a_0 is the Bohr radius. Thus the Van der Waals energy is

$$E_{VdW}(R) \simeq -\frac{6e^2 a_0^5}{R^6}.$$