PHYSICS 828

Home Work Assignment # 3 1/21/2011

Due: Fri., Jan. 28, 2011.

Completed assignments should be placed in the grader N. Ramalingam's mail box in PRB by 5:00 PM.

1. Consider the *2D isotropic harmonic oscillator:*

$$H = \frac{1}{2\mu}\left(P_x^2 + P_y^2\right) + \frac{1}{2}\mu\omega^2\left(X^2 + Y^2\right).$$

You have already solved the 2D anisotropic oscillator in Cartesian coordinates in Ex. 10.2.2. Read Shankar Ex. 12.3.7 (p. 316 - 317) parts (1) through (10) carefully and understand the differential equation approach outlined in that problem; you do *not* need to hand in the solution. Instead, you should solve the problem using the following algebraic approach.

(a) Define

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{X}{\ell} + i\frac{\ell P_x}{\hbar}\right), \qquad a_y = \frac{1}{\sqrt{2}}\left(\frac{Y}{\ell} + i\frac{\ell P_y}{\hbar}\right)$$

where the length $\ell = \sqrt{\hbar/\mu\omega}$.
Rewrite H in terms of $a_x, a_x^\dagger, a_y, a_y^\dagger$ and write down the commutation relations between the four operators.

(b) Express the angular momentum $L_z = XP_y - YP_x$ in terms of $a_x, a_x^\dagger, a_y, a_y^\dagger$. (Note that while H has a very simple form in terms of these operators, L_z does not.)
Show using the algebra of these operators that $[H, L_z] = 0$ as you would expect for an isotropic 2D oscillator.

(c) Define "left" and "right circular" operators

$$b_L = \frac{1}{\sqrt{2}}\left(a_x + ia_y\right), \qquad b_R = \frac{1}{\sqrt{2}}\left(a_x - ia_y\right).$$

Show that the only non-zero commutators between the operators $b_L, b_L^\dagger, b_R, b_R^\dagger$ are $\left[b_L, b_L^\dagger\right] = \left[b_R, b_R^\dagger\right] = 1$.

(d) Show that both the Hamiltonian H and the angular momentum L_z can be written very simply in terms of the number operators $N_L = b_L^\dagger b_L$ and $N_R = b_R^\dagger b_R$ for "left" and "right circular" quanta. (Note that we have managed to obtain a very simple form for L_z while maintaining that of H.)

(e) Show that common eigenstates of H and L_z can be written as

$$|n_R, n_L\rangle = \frac{\left(b_R^\dagger\right)^{n_R}\left(b_L^\dagger\right)^{n_L}}{\sqrt{n_R! n_L!}}|0,0\rangle$$

and find the corresponding eigenvalues.

(f) Show that an energy eigenvalue $(n+1)\hbar\omega$ has an $(n+1)$-fold degeneracy corresponding to angular momentum $m\hbar$ with $m = -n, -n+2, \ldots, n-4, n-2, n$. Also argue that for a given value of n (energy) and of m (angular momentum), there is a unique eigenstate.

2. *Landau levels for a charged particle in an external magnetic field:* Shankar Ex. 12.3.8 (p. 300 - 318).

3. *Read* Shankar Ex. 12.4.3 (p. 320). You don't need to turn it in, but it might help you with the next question.

4. (a) Show that the 3D rotation matrices $\mathcal{R}_{\hat{\mathbf{n}}}(\epsilon)$ for a counter-clockwise rotation by an infinitesimal angle ϵ about an axis $\hat{\mathbf{n}}$ are given by:

$$\mathcal{R}_{\hat{\mathbf{x}}}(\epsilon_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\epsilon_x \\ 0 & \epsilon_x & 1 \end{pmatrix}, \quad \mathcal{R}_{\hat{\mathbf{y}}}(\epsilon_y) = \begin{pmatrix} 1 & 0 & \epsilon_y \\ 0 & 1 & 0 \\ -\epsilon_y & 0 & 1 \end{pmatrix},$$

$$\text{and} \quad \mathcal{R}_{\hat{\mathbf{z}}}(\epsilon_z) = \begin{pmatrix} 1 & -\epsilon_z & 0 \\ \epsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By "show", I mean that at the very least you should draw some simple 2D pictures of the plane perpendicular to the axis of rotation (in each case) and give a geometrical argument. Make sure you understand the $\pm$ signs.

(b) Check that rotations in 3D do not commute by showing that

$$\mathcal{R}_{\hat{\mathbf{y}}}(-\epsilon_y)\mathcal{R}_{\hat{\mathbf{x}}}(-\epsilon_x)\mathcal{R}_{\hat{\mathbf{y}}}(\epsilon_y)\mathcal{R}_{\hat{\mathbf{x}}}(\epsilon_x) = \mathcal{R}_{\hat{\mathbf{z}}}(-\epsilon_x\epsilon_y).$$

(c) Let $U_{\hat{\mathbf{n}}}(\epsilon)$ be the unitary operator representing the effect of the rotation $\mathcal{R}_{\hat{\mathbf{n}}}(\epsilon)$ on the Hilbert space of states of a quantum system. Thus the quantum operators must satisfy the relation

$$U_{\hat{\mathbf{y}}}(-\epsilon_y)U_{\hat{\mathbf{x}}}(-\epsilon_x)U_{\hat{\mathbf{y}}}(\epsilon_y)U_{\hat{\mathbf{x}}}(\epsilon_x) = U_{\hat{\mathbf{z}}}(-\epsilon_x\epsilon_y).$$

Using $U_{\hat{\mathbf{n}}}(\epsilon) = \mathbf{1} - i\epsilon L_{\mathbf{n}}/\hbar$ where $\mathbf{n} = \mathbf{x}, \mathbf{y}, \mathbf{z}$, and $L_{\mathbf{n}} = \mathbf{L} \cdot \hat{\mathbf{n}}$, derive the commutation relation

$$[L_x, L_y] = i\hbar L_z.$$

5. Consider angular momentum operators $\mathbf{J} = (J_x, J_y, J_z)$, that obey the standard algebra $[J_x, J_y] = i\hbar J_z$ and cyclic permutations.
Let $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$
Show that
(a) $[J_z, J_+] = \hbar J_+$
(b) $[J_z, J_-] = -\hbar J_-$
(c) $[J_+, J_-] = 2\hbar J_z$
(d) $\left[J^2, J_+\right] = \left[J^2, J_-\right] = \left[J^2, J_z\right] = 0$
(e) $J^2 = \frac{1}{2}\left(J_+J_- + J_-J_+\right) + J_z^2$