<u>PHYSICS 880.06</u>

Home Work Assignment # 5 10/25/2011

<u>Due:</u> Tue., Nov. 1, 2011.

Problem: *Optical conductivity of the free electron gas with impurities:*

In this problem you will consider a simple model of the optical conductivity of a metal and explore its consequences. Ashcroft and Mermin give an elementary discussion in Chapter 1 that may be worth reviewing before you do this problem.

The non-equilibrium distribution function $f(\mathbf{r}, \mathbf{k}, t)$ for electrons is obtained by solving the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v_k} \cdot \nabla_{\mathbf{r}} f + \frac{e\mathbf{E}}{\hbar} \cdot \nabla_{\mathbf{k}} f = -\frac{[f - f^0]}{\tau}$$

in the relaxation time approximation. Here $\mathbf{v_k} = \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}} / \hbar$ is the velocity, f^0 is the equilibrium distribution and τ the scattering time from impurities. We consider weak disorder: $k_F \ell \sim \epsilon_F \tau \gg 1$.

(1) Consider an external electric field $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t)$. Because of the large value of c we can ignore the spatial variation of the field and just focus on its time-dependence in solving the Boltzmann equation.
Linearize the Boltzmann equation in the small field limit and show that the distribution function is given by

$$f = f^0 + \frac{e\tau \mathbf{v_k} \cdot \mathbf{E_0}}{1 - i\omega\tau} \left(\frac{-\partial f^0}{\partial \epsilon} \right) e^{-i\omega t}.$$

(2) Show that the frequency dependent conductivity is given by

$$\sigma(\omega) = \sigma(0) \frac{1 + i\omega\tau}{1 + \omega^2 \tau^2}$$

where $\sigma(0)$ is the static (or d.c.) conductivity discussed in class. You can simplify the problem here onwards by choosing a parabolic dispersion $\epsilon_{\mathbf{k}}$.

(3) The optical conductivity is complex: $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$. Show that its real part σ_1, that is related to absorption of energy, satisfies the sum rule

$$\int_0^\infty d\omega \sigma_1(\omega) = \frac{\omega_p^2}{8}.$$

Here $\omega_p^2 = 4\pi n e^2/m$ is the *plasma frequency* (n = density of electrons, m = electron mass), whose physical meaning will be clarified below.
This sum rule is a very general constraint on the optical conductivity of *any* system (with disorder, interactions, ...). Our calculation only serves to illustrate that the result of part (b) satisfies this general sum rule.

(4) Make a careful sketch of $\sigma_1(\omega)$. Why is the area under this curve *independent* of the scattering time τ and only dependent on the density n of electrons?

(5) Let us now understand the propagation of e.m. radiation in a metal. Following the derivation on p. 17 of Ashcroft and Mermin, show that the (complex) dielectric function $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$ of a metal is related to its optical conductivity via

$$\epsilon(\omega) = 1 + i\frac{4\pi}{\omega}\sigma(\omega).$$

If we choose a solution of Maxwell's equation of the form
$\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{K}\cdot\mathbf{r} - \omega t)]$ then show that the *complex* K is given by

$$K = \sqrt{\epsilon(\omega)}\,\frac{\omega}{c} \equiv (n_1 + i n_2)\frac{\omega}{c}.$$

What is the physical meaning of a complex K?

(6) We will assume (without proof) the following result from classical electrodynamics (which is actually quite easy to prove). The *reflection coefficient* R for radiation incident upon the interface between vacuum and a material with $\epsilon(\omega) = (n_1 + i n_2)^2$ is given by

$$R = \frac{(n_1 - 1)^2 + n_2^2}{(n_1 + 1)^2 + n_2^2}.$$

Using the results derived above show that in the low frequency regime
$\omega \ll 1/\tau \ll \omega_p$, one has

$$R \simeq 1 - 2\left(\frac{2\omega}{\omega_p^2\tau}\right)^{1/2}.$$

A reflection coefficient close to unity is the reason why metals are shiny!

(7) Show that in the high frequency regime $\omega \gg 1/\tau$ the real part of the dielectric function ϵ_1 is given by

$$\epsilon_1(\omega) = 1 - \frac{\omega_p^2}{\omega^2}.$$

The plasma frequency is now seen as the frequency at which ϵ_1 changes sign. Using n_1 and n_2 defined above, argue that for $\omega > \omega_p$ a metal becomes transparent. This phenomenon that is called "ultraviolet transparency" in view of the values of metallic plasma frequencies (see Ashcroft and Mermin, p. 18).

Finally, read Ashcroft and Mermin (p. 19 - 20) where they discuss collective oscillations in a metal at the plasma frequency.