1 The disc amplitude

Note: The treatment here follows Zwiebach.

Suppose we have two open strings scattering off each other. This will be a 4-point diagram, since there are two incoming quanta and two outgoing quanta. We will take the open strings to be in their ground states, which are tachyons with

$$m^2 = -p^2 = -\frac{1}{\alpha'} \tag{1}$$

We will have

$$p_1 + p_2 + p_3 + p_4 = 0 \tag{2}$$

The interaction will be described by inserting the vertex operators for the tachyons on the boundary of a disc. For vertex operators

$$e^{ip_iX} \tag{3}$$

the OPE is

$$e^{ip_iX}(z)e^{ip_jX}(z') \sim |z - z'|^{2\alpha' p_i \cdot p_j} e^{i(p_i+p_j)X} + \ldots \tag{4}$$

The general correlation function is

$$< \prod_i e^{p_iX}(z_i) >= \prod_{i<j} |z_i - z_j|^{2\alpha' p_i \cdot p_j} \tag{5}$$

We would need to integrate over the locations of the insertions, which we study now.

2 Disc as an upper half plane (UHP)

Let the disc be defined as

$$|w| < 1 \tag{6}$$

Consider the analytic map

$$z = i\frac{1-w}{1+w} \tag{7}$$

We can see that this maps the disc to the upper half plane. The boundary of the disc goes to the real axis. This can be seen by noting that the boundary is

$$w = e^{i\theta}, \quad -\pi \le \theta < \pi \tag{8}$$

But

$$i\frac{1-e^{i\theta}}{1+e^{i\theta}} = i\frac{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} = i\frac{-2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = -\tan\frac{\theta}{2} \tag{9}$$

Thus we get real z, and these real numbers cover the real line (including the point at infinity) once. Thus the boundary of the disc maps to the boundary of the UHP. Now we check that the interior of the disc maps to the interior of the UHP. We need to show that the imaginary part of z is positive. Let $w = x + iy$. We have

$$z = i\frac{(1-x)-iy}{(1+x)+iy} = i\frac{[(1-x)-iy][(1+x)-iy]}{(1+x)^2+y^2} = i\frac{(1-x^2-y^2)-2i(1-x)y}{(1+x)^2+y^2} \tag{10}$$

Using $x^2+y^2<1$, we see that $Im(z)>0$, so we are in the UHP.

We now check that the UHP is preserved by fractional linear transformations

$$z' = \frac{az+b}{cz+d}, \quad ad-bc=1 \tag{11}$$

where a, b, c, d are real. Note that the real line goes to the real line. We could have scaled all of a, b, c, d by a uniform constant, and the transformation would not change. In this way we can have $ab-cd=\lambda$ for any $\lambda>0$, but note that we cannot get $\lambda<0$. Now let us check that we remain in the UHP. Let $z=x+iy$. We have

$$z' = \frac{(ax+b)+iay}{(cx+d)+icy} = \frac{[(ax+b)+iay][(cx+d)-icy]}{(cx+d)^2+c^2y^2} \tag{12}$$

Thus

$$Im[z'] = y\frac{ad-bc}{(cx+d)^2+c^2y^2} = \frac{y}{(cd+d)^2+c^2y^2} > 0 \tag{13}$$

so we are in the UHP.

Let us see how variables transform under the fractional linear transformation. We have

$$dz' = [\frac{adz}{cz+d} - \frac{(az+b)c}{(cz+d)^2}]dz = \frac{a(cz+d)-c(az+b)}{(cz+d)^2}dz = \frac{ad-bc}{(cz+d)^2}dz = \frac{1}{(cz+d)^2}dz \tag{14}$$

$$z_1'-z_2' = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} = \frac{(az_1+b)(cz_2+d)-(az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)} = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} = \frac{z_1-z_2}{(cz_1+d)(cz_2+d)} \tag{15}$$

3 Writing the amplitude

We can use fractional linear transformations to fix three points to arbitrary values. It is conventional to fix z_1, z_3, z_4. Then $z_2 \equiv z$ will need to be integrated over. We have to decide what measure to use for z. The goal is to get invariance under fractional linear transformations. It turns out that the correct measure is

$$d\mu = dz|z_1-z_3||z_3-z_4||z_4-z_1| \tag{16}$$

This will transform to

$$\frac{dz}{|cz+d|^2}\frac{|z_1-z_3|}{|(cz_1+d)(cz_3+d)|}\frac{|z_3-z_4|}{|(cz_3+d)(cz_4+d)|}\frac{|z_4-z_1|}{|(cz_4+d)(cz_1+d)|} = \frac{dz}{|cz+d|^2}\frac{|z_1-z_3||z_3-z_4||z_4-z_1|}{|(cz_1+d)|^2|(cz_3+d)|^2|(cz_4+d)|^2} \tag{17}$$

The actual correlator of vertex operators has the factor

$$|z-z_1|^{2\alpha' p_2\cdot p_1}|z-z_3|^{2\alpha' p_2\cdot p_3}|z-z_4|^{2\alpha' p_2\cdot p_4}|z_1-z_3|^{2\alpha' p_1\cdot p_3}|z_3-z_4|^{2\alpha' p_3\cdot p_4}|z_4-z_1|^{2\alpha' p_4\cdot p_1} \quad (18)$$

This transforms to itself times the multiplicative factor

$$\begin{aligned} &|(cz+d)|^{-2\alpha' p_2\cdot p_1-2\alpha' p_2\cdot p_3-2\alpha' p_2\cdot p_4} \\ &|(cz_1+d)|^{-2\alpha' p_2\cdot p_1-2\alpha' p_1\cdot p_3-2\alpha' p_1\cdot p_4} \\ &|(cz_3+d)|^{-2\alpha' p_2\cdot p_3-2\alpha' p_3\cdot p_1-2\alpha' p_3\cdot p_4} \\ &|(cz_4+d)|^{-2\alpha' p_2\cdot p_4-2\alpha' p_4\cdot p_3-2\alpha' p_4\cdot p_3} \end{aligned} \quad (19)$$

Thus the total power of $|cz+d|$ is

$$-2[1+\alpha' p_2\cdot(p_1+p_3+p_4)] = -2[1-\alpha' p_2\cdot p_2] = -2[1-\alpha'\frac{1}{\alpha'}] = 0 \quad (20)$$

and similarly for the other factors. Thus the integral is invariant under fractional linear transformations.

We now set

$$z_1 = 0, \quad z_3 = 1, \quad z_4 = \infty \quad (21)$$

The integral now simplifies to

$$\int dz|z_4|^2|z|^{2\alpha' p_2\cdot p_1}|z-1|^{2\alpha' p_2\cdot p_3}|z_4|^{2\alpha' p_2\cdot p_4}|z_4|^{2\alpha' p_3\cdot p_4}|z_4|^{2\alpha' p_4\cdot p_1} \quad (22)$$

The power of $|z_4|$ again cancels

$$2[1+\alpha' p_4\cdot(p_1+p_2+p_3)] = 2[1-\alpha' p_4\cdot p_4] = 0 \quad (23)$$

and we get

$$\int dz|z|^{2\alpha' p_2\cdot p_1}|z-1|^{2\alpha' p_2\cdot p_3} \quad (24)$$

The variable z lies on the real line between z_1 and z_3, which gives $0 < z < 1$. Thus we have

$$\int_0^1 dz z^{2\alpha' p_2\cdot p_1}(1-z)^{2\alpha' p_2\cdot p_3} \quad (25)$$

4 Converting to physical variables

Note that

$$s = -(p_1+p_2)^2 = -\frac{2}{\alpha'} - 2p_{1\cdot 2} \quad (26)$$

so that

$$2\alpha' p_1\cdot p_2 = -\alpha' s - 2 = -(\alpha' s+1) - 1 \quad (27)$$

Similarly,

$$t = -(p_2+p_3)^2 = -\frac{2}{\alpha'} - 2p_{2\cdot 3} \quad (28)$$

so that

$$2\alpha' p_2 \cdot p_3 = -\alpha' t - 2 = -(\alpha' t + 1) - 1 \tag{29}$$

We write

$$\alpha(s) = \alpha' s + 1, \quad \alpha(t) = \alpha' t + 1 \tag{30}$$

Then the integral becomes

$$\int_0^1 dz z^{-\alpha(s)-1}(1-z)^{-\alpha(t)-1} \tag{31}$$

Recall that the Beta function is defined as

$$B[a,b] = \int_0^1 dx x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{32}$$

Thus the integral equals

$$\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{33}$$

The Γ function has poles when its argument is zero or a negative integer. Thus there are poles for

$$\alpha(s) = n, \quad n = 0, 1, \ldots \tag{34}$$

and for

$$\alpha(t) = n, \quad n = 0, 1, \ldots \tag{35}$$

Thus the poles in the s-channel occur for

$$\alpha' s + 1 - n, \quad s = \frac{1}{\alpha'}(n-1), \quad n = 0, 1, \ldots \tag{36}$$

We see that these are just the masses of the open string states. The same poles occur in the t-channel.

5 Closed strings

We can do a similar analysis for closed strings, and the amplitude is called the Virasoro-Shapiro amplitude.. This time the world sheet is a sphere, and we have 4 points on this sphere. By conformal invariance, we can again set three points to $0, 1, \infty$. The last point is z.

The closed string tachyons have

$$p^2 = -m^2 = \frac{4}{\alpha'} \tag{37}$$

The fractional linear transformations are now

$$z' = \frac{az+b}{cz+d} \tag{38}$$

with all numbers complex. We have

$$d^2z \to d^2z \frac{1}{(cz+d)^2(\bar{c}\bar{z}+\bar{d})^2} = dd\bar{z}\frac{1}{|cz+d|^4} \tag{39}$$

The measure will now be

$$d\mu = d^2z|z_1 - z_3|^2|z_3 - z_4|^2|z_4 - z_1|^2 \tag{40}$$

The amplitude for 4 vertex operators will be

$$\int d^2z|z_1-z_3|^2|z_3-z_4|^2|z_4-z_1|^2|z-z_1|^{\alpha' p_2\cdot p_1}|z-z_3|^{\alpha' p_2\cdot p_3}|z-z_4|^{\alpha' p_2\cdot p_4}|z_1-z_3|^{\alpha' p_1\cdot p_3}|z_3-z_4|^{\alpha' p_3\cdot p_4}|z_4-z_1|^{\alpha' p_4\cdot p_1} \tag{41}$$

For the positions of the operators chosen, we get

$$\int d^2z|z_4|^4|z|^{4\alpha' p_2\cdot p_1}|z-1|^{\alpha' p_2\cdot p_3}|z_4|^{\alpha' p_2\cdot p_4}|z_4|^{\alpha' p_3\cdot p_4}|z_4|^{\alpha' p_4\cdot p_1} \tag{42}$$

The power of z_4 cancels, and we get

$$\int d^2z|z|^{\alpha' p_2\cdot p_1}|z-1|^{\alpha' p_2\cdot p_3} \tag{43}$$

We have

$$\Gamma[A] = \int_0^\infty dt t^{A-1}e^{-t} \tag{44}$$

We have

$$\int_0^\infty dt t^{A-1}e^{-|\mu|^2 t} = \frac{1}{|\mu|^{2A}}\int_0^\infty d(t|\mu|^2)(|\mu|^2 t)^{A-1}e^{-|\mu|^2 t} = |\mu|^{-2A}\Gamma[A] \tag{45}$$

Thus

$$\frac{1}{|\mu|^{2A}} = \frac{1}{\Gamma[A]}\int_0^\infty dt t^{A-1}e^{-|\mu|^2 t} \tag{46}$$

The quantity that we wish to compute is

$$\mathcal{A} = \int d^2z|z|^{\alpha' p_2\cdot p_1}|z-1|^{\alpha' p_2\cdot p_3} \tag{47}$$

Now use (46) with

$$-2A = \alpha' p_2\cdot p_1, \quad A = -\frac{1}{2}\alpha' p_2\cdot p_1 \tag{48}$$

to get

$$|z|^{\alpha' p_2\cdot p_1} = \frac{1}{\Gamma[-\frac{1}{2}\alpha' p_2\cdot p_1]}\int_0^\infty dt t^{-\frac{1}{2}\alpha' p_2\cdot p_1 - 1}e^{-|z|^2 t} \tag{49}$$

Similarly,

$$|z-1|^{\alpha' p_2\cdot p_3} = \frac{1}{\Gamma[-\frac{1}{2}\alpha' p_2\cdot p_3]}\int_0^\infty ds s^{-\frac{1}{2}\alpha' p_2\cdot p_3 - 1}e^{-|z-1|^2 s} \tag{50}$$

Writing $z = x + iy$, we have

$$\mathcal{A} = \frac{1}{\Gamma[-\frac{1}{2}\alpha' p_2\cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2\cdot p_3]}\int_{-\infty}^\infty\int_{-\infty}^\infty dxdy\int_0^\infty dt\int_0^\infty ds t^{-\frac{1}{2}\alpha' p_2\cdot p_1 - 1}s^{-\frac{1}{2}\alpha' p_2\cdot p_3 - 1}e^{-|z|^2 t}e^{-|z-1|^2 s} \tag{51}$$

We have

$$e^{-|z|^2 t}e^{-|z-1|^2 s} = e^{-(x^2+y^2)t-((x-1)^2+y^2)s} = e^{-s}e^{-(t+s)x^2+2sx}e^{-(t+s)y^2} \tag{52}$$

We have

$$\int_0^\infty dx e^{-(t+s)x^2+2sx} = \sqrt{\frac{\pi}{t+s}} e^{\frac{s^2}{t+s}} \tag{53}$$

$$\int_0^\infty dy e^{-(t+s)y^2} = \sqrt{\frac{\pi}{t+s}} \tag{54}$$

Thus we get

$$\mathcal{A} = \frac{1}{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_3]} \int_0^\infty dt \int_0^\infty ds t^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - 1} s^{-\frac{1}{2}\alpha' p_2 \cdot p_3 - 1} e^{-s} \frac{\pi}{t+s} e^{\frac{s^2}{t+s}} \tag{55}$$

Write

$$t = au, \quad s = (1-a)u \tag{56}$$

Thus

$$t + s = u \tag{57}$$

We have

$$\frac{\partial t}{\partial a} = u, \quad \frac{\partial t}{\partial u} = a, \quad \frac{\partial s}{\partial a} = -u, \quad \frac{\partial s}{\partial u} = 1 - a \tag{58}$$

$$\frac{\partial(t,s)}{\partial(a,u)} = u(1-a) + au = u \tag{59}$$

Thus we get

$$\mathcal{A} = \frac{1}{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_3]} \int_0^1 da \int_0^\infty du u (au)^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - 1} ((1-a)u)^{-\frac{1}{2}\alpha' p_2 \cdot p_3 - 1} e^{-(1-a)u} \frac{\pi}{u} e^{\frac{(1-a)^2 u^2}{u}} \tag{60}$$

$$\mathcal{A} = \frac{\pi}{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_3]} \int_0^1 da \int_0^\infty du (a)^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - 1} (1-a)^{-\frac{1}{2}\alpha' p_2 \cdot p_3 - 1} (u)^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - \frac{1}{2}\alpha' p_2 \cdot p_3 - 2} e^{-a(1-a)u} \tag{61}$$

First we do the u integral

$$\int_0^\infty du (u)^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - \frac{1}{2}\alpha' p_2 \cdot p_3 - 2} e^{-a(1-a)u} = \Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1 - \frac{1}{2}\alpha' p_2 \cdot p_3 - 1](a(1-a))^{\frac{1}{2}\alpha' p_2 \cdot p_1 + \frac{1}{2}\alpha' p_2 \cdot p_3 + 1} \tag{62}$$

The a integral now is

$$\int_0^1 da (a)^{-\frac{1}{2}\alpha' p_2 \cdot p_1 - 1} (1-a)^{-\frac{1}{2}\alpha' p_2 \cdot p_3 - 1} (a(1-a))^{\frac{1}{2}\alpha' p_2 \cdot p_1 + \frac{1}{2}\alpha' p_2 \cdot p_3 + 1} = \int_0^1 da (a)^{\frac{1}{2}\alpha' p_2 \cdot p_3} (1-a)^{\frac{1}{2}\alpha' p_2 \cdot p_1} \tag{63}$$

This gives

$$\int_0^1 da (a)^{\frac{1}{2}\alpha' p_2 \cdot p_3} (1-a)^{\frac{1}{2}\alpha' p_2 \cdot p_1} = \frac{\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_3 + 1]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + 1]}{\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + \frac{1}{2}\alpha' p_2 \cdot p_3 + 2]} \tag{64}$$

Thus overall we get

$$\mathcal{A} = \pi \frac{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1 - \frac{1}{2}\alpha' p_2 \cdot p_3 - 1]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_3 + 1]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + 1]}{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_3]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + \frac{1}{2}\alpha' p_2 \cdot p_3 + 2]} \tag{65}$$

Note that

$$-\frac{1}{2}\alpha'(p_2 \cdot p_1 + p_2 \cdot p_3 + 4) = -\frac{1}{2}\alpha'(p_2 \cdot p_1 + p_2 \cdot p_3 + p_2 \cdot p_2) = \frac{1}{2}\alpha' p_2 \cdot p_4 \tag{66}$$

Thus

$$\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1 - \frac{1}{2}\alpha' p_2 \cdot p_3 - 1] = \Gamma[\frac{1}{2}\alpha' p_2 \cdot p_4 + 1] \tag{67}$$

and

$$\mathcal{A} = \pi \frac{\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_4 + 1]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_3 + 1]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + 1]}{\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_1]\Gamma[-\frac{1}{2}\alpha' p_2 \cdot p_3]\Gamma[\frac{1}{2}\alpha' p_2 \cdot p_1 + \frac{1}{2}\alpha' p_2 \cdot p_3 + 2]} \tag{68}$$

Let us convert this to physical variables. We have

$$s = -(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = -\frac{8}{\alpha'} - 2p_1 \cdot p_2 \tag{69}$$

Thus

$$\alpha' p_1 \cdot p_2 = -\frac{1}{2}(\alpha' s + 8) = -\frac{1}{2}(\alpha' s + 4) - 2 \tag{70}$$

Writing

$$\alpha(s) = \frac{1}{4}(\alpha' s + 4) \tag{71}$$

we have

$$\frac{1}{2}\alpha' p_1 \cdot p_2 + 1 = -\alpha(s) \tag{72}$$

Thus we get

$$\mathcal{A} = \pi \frac{\Gamma[-\alpha(u)]\Gamma[-\alpha(t)]\Gamma[-\alpha(s)]}{\Gamma[1+\alpha(s)]\Gamma[1+\alpha(t)]\Gamma[1+\alpha(u)]} \tag{73}$$

The s channel poles are at

$$\alpha(s) = n, \quad n = 0, 1, 2, \ldots \tag{74}$$

$$\frac{1}{4}(\alpha' s + 4) = n, \quad s = \frac{4}{\alpha'}(n-1), \quad n = 0, 1, \ldots \tag{75}$$

which agrees with the masses of the string states.