1 Path integrals

The string worldsheet was 2-dimensional. On this worldsheet we had the variables $X^\mu(\tau, \sigma)$. Upon quantization we have to do a path integral over these variables. We can also regard the theory in a Hamiltonian language, where X^μ will become local operators.

More general string theories can be made by taking again the 2-d string worldsheet (τ, σ) and then making some other suitable field theory on this worldsheet. To understand such field theories in generality, let us review path integrals and their Hamiltonian description.

2 The Ising model

Let us start with a very simple system. Instead of the 2-d worldsheet, just consider a 1-d line, parametrized by the variable τ. We further assume that this line is discretized to a set of lattice points k, $k = 1, \ldots N$. Instead of a field variable like X^μ which can equal any real number, we allow each point lattice point to carry only two possibilities; we call these spin up and spin down. Note that there are

$$\# = 2^N \tag{1}$$

possible configurations for the spins on the lattice.

To describe the dynamics we have to give the action for any configuration of spins. We will let the action be *local*: thus the total action will be the sum of contributions from neighboring pairs of spins. If both members are up, we will have action A, if both down we will have B, and if one is up and one down, we will have C. Note that we have taken the action to be symmetric under reversal of the direction of τ: for this case of one spin up and one down the action does not depend on which of the spins was up and which was down.

Thus for any given configuration of spins we have an action

$$S = \sum_{i=1}^{N-1} S_i \tag{2}$$

where S_i is the action from the interval between the i and $i+1$ lattice sites. The total partition function of this statistical system is

$$Z = \sum_{conf} e^{-S} \tag{3}$$

where $\sum_{conf}$ is the sum over the 2^N possible configurations of spins. The question now is: how so we compute such a partition function?

Let us write the configuration at lattice site i as a 2-component vector. Thus we have

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv V_1 \qquad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv V_2 \tag{4}$$

We encode the action from the interval $(i, i+1)$ into a matrix

$$M = \begin{pmatrix} e^{-A} & e^{-C} \\ e^{-C} & e^{-B} \end{pmatrix} \tag{5}$$

Suppose the state at site i is V_a and at site $i+1$ is V_b, where $a, b = 1, 2$. Then the contribution to e^{-S} from the interval $(i, i+1)$ is given by

$$e^{-S} \to (V_b)^T M V_a \tag{6}$$

So far we have not learnt much by this formalism. The interesting part comes now. Suppose we want the contribution from the interval $(i, i+2)$. The contribution to e^{-S} is now the product of contributions from two different slices, $(i, i+1)$ and $(i+1, i+2)$. Suppose we have the state 1 at i, the state 1 at $i+1$, and the state 2 at $i+2$. Then the contribution to e^{-S} from $(i, i+2)$ is

$$e^{-S} \to M_{21} M_{11} \tag{7}$$

If the middle spin was 2 instead, we would get

$$e^{-S} \to M_{22} M_{21} \tag{8}$$

Thus summing over the two possibilities of the middle spin is equivalent to multiplying the two M matrices: we sum over the index common to these matrices. We see that if the configuration at i is V^i and at $i+2$ is V^{i+2} then we get

$$e^{-S} = (V^{i+2})^T M M V^i \tag{9}$$

Now consider the entire chain of spins, and its total path integral. There are different kinds of boundary conditions that we can put at the ends of the chain:

(a) We can specify the spin state at $i = 1$, $i = N$ by giving vectors

$$V^1, \quad V^N \tag{10}$$

which specify the value of the spin at these ends. In this case

$$Z = (V^N)^T M^{N-1} V^1 \tag{11}$$

(b) We take 'free' boundary conditions at the ends; i.e., the spins at the ends have to be summed over both possibilities with equal weight, just as we do at all other lattice sites. Then it can be seen that we should define the vector

$$V_F = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{12}$$

and then

$$Z = V_F^T M^{N-1} V_F \tag{13}$$

(c) We can join the two ends of our lattice to make a loop with no free ends. Thus we can identify site N with site 1. There are $N-1$ intervals in the loop which contribute to the action and we have

$$Z = Tr\ [M^{N-1}] \tag{14}$$

Of course in the cases where we do not have a loop we can choose different boundary conditions at the two ends, and use the appropriate vectors at these ends.

Let us now move to the next step: how do we compute a quantity like (11)? The matrix M was symmetric because of time reversal symmetry $\tau \to -\tau$. Thus it can be diagonalized by an orthogonal transformation

$$O^T M O = D \tag{15}$$

where $O^T O = O O^T = I$ and

$$D = \begin{pmatrix} e^{-\lambda_1} & 0 \\ 0 & e^{-\lambda_2} \end{pmatrix} \tag{16}$$

is a diagonal matrix. Then we can write

$$Z = V_N^T M^{N-1} V - 1 = V_N^T O O^T M O O^T M \ldots O^T M O O^T V_1 = V_N'^T D^{N-1} V_1' \tag{17}$$

where

$$V_1' = O^T V_1 = \begin{pmatrix} (V_1')_1 \\ (V_1')_2 \end{pmatrix}, \quad V_N' = O^T V_N = \begin{pmatrix} (V_N')_1 \\ (V_N')_2 \end{pmatrix} \tag{18}$$

Thus all we have to do is find O for the matrix M. Then we can find the vectors V' from the given vectors V. Then the partition function is given by

$$Z = (V_N')_1 (V_1')_1 e^{-(N-1)\lambda_1} + (V_N')_2 (V_1')_2 e^{-(N-1)\lambda_2} \tag{19}$$

So it does not matter how large N is, we can still find Z with the same amount of effort. The matrix M is called the 'transfer matrix' for the problem.

But one thing we realize from this method of solution is that it is useful to make 'formal linear combinations' of allowed configurations. Thus the spin at $i=1$ could be either up or down; there was no in-between configuration. Thus the only allowed configurations gave the vectors (4). But we found it useful to make the vector V_1', which had entries which were arbitrary *real* numbers in general. This vector is then a linear superposition of configurations, just like we take linear superpositions of states in quantum mechanics. In fact we have here the Hamiltonian description of the path integral that we started with. The vectors V_1', V_N' are the states at the initial and final time slices, and $-\log D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is the Hamiltonian. In the original basis, the states were V_n, V_1, and $-\log M$ was the Hamiltonian.

3 Some formal relations

Consider the 2-point correlation function of a free scalar field, in D Euclidean dimensions. The scaling dimension of the field is found by asking that the action have no units. Thus we get

$$S = \frac{1}{2}\int d^D\xi \partial\phi\partial\phi \to L^{D-2}[\phi]^2 = 1 \tag{20}$$

which gives

$$[\phi] = L^{-\frac{(D-2)}{2}} \tag{21}$$

Thus in 4-d we have ϕ with units of $\frac{1}{L}$ which is units of mass. But in 2-d ϕ seems to have no units at all. We will see however that the units of ϕ in 2-d are slightly ill defined.

If ϕ has units as above, then we expect that the 2-point function of ϕ will behave as

$$< \phi(x)\phi(0) >= \frac{1}{|x|^{D-2}} \tag{22}$$

In 2-d we see that this is a constant function. This does not look right, since it would imply that the correlation does not drop with distance at all. More precisely, we can ask that the 2-point function satisfy the field equation

$$\triangle_x < \phi(x)\phi(0) >= 0, \quad x \neq 0 \tag{23}$$

Thus in 2-d we should have

$$< \phi(x)\phi(0) >\sim -\ln|x| \tag{24}$$

We can fix the sign in this equation by noting that the correlation function should be positive and should grow for small distances.

Let us now fix coefficients in this expression. The path integral is

$$Z = \int d\phi e^{-\frac{1}{2}\partial\phi\partial\phi} = \int d\phi e^{\frac{1}{2}\phi\triangle\phi} \tag{25}$$

We can add a source

$$Z = \int d\phi e^{\frac{1}{2}\phi\triangle\phi + J\phi} \tag{26}$$

The path integral over the Gaussian will give

$$Z = e^{-\frac{1}{2}J\triangle^{-1}J} \tag{27}$$

where we have used the general formula for a Gaussian integral

$$\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \tag{28}$$

The 2-point function is

$$< \phi(x)\phi(0) >= \frac{1}{Z}\frac{\delta Z}{\delta J(x)}\frac{\delta Z}{\delta J(0)} = -\triangle^{-1}(x,0) \tag{29}$$

(The factor of 2 disappears because the first derivative can act on either of the two Js in the exponent.) Thus

$$\triangle < \phi(x)\phi(0) >= -\delta(x) \tag{30}$$

Let us now find $\triangle^{-1}$ in 2-d. We write

$$\triangle \ln r = \mu\delta(0) \tag{31}$$

and we find μ. Let us integrate both sides over a circular disc of radius R around the origin. The LHS gives, using the Stokes theorem

$$\int d^2x \triangle(\ln r) = \int dl \partial_r(\ln r) = \int_R dl \frac{1}{R} = 2\pi R \frac{1}{R} = 2\pi \tag{32}$$

where $\int_R dl$ is the integral over the circular boundary of the disc of radius R. The RHS of (31) gives μ Thus we have

$$\triangle(\ln r) = 2\pi\delta(0) \tag{33}$$

Thus from (30) we should have

$$< \phi(x)\phi(0) >= -\frac{1}{2\pi} \ln |x| \tag{34}$$

But this cannot be quite right, since x has units of length, and we cannot take its log. Thus there must be a length scale that appears to make x dimensionless. But in a conformal theory there is no length scale! Thus we see that the field ϕ is not a good conformal field at all, and cutoffs used in the definition of the path integral will play a role in its exact correlation function. We will look at these cutoffs later, but for now we write the correlation function as

$$< \phi(x)\phi(0) >= -\frac{1}{2\pi} \ln \frac{|x|}{a} \tag{35}$$

where a has units of length.

In the problem that we have the action is actually

$$S = T \int \frac{1}{2} \partial X \partial X = \frac{1}{2\pi\alpha'} \int \frac{1}{2} \partial X \partial X \tag{36}$$

We can make the dimensionless field

$$\phi = \frac{X}{\sqrt{2\pi\alpha'}} \tag{37}$$

Then we get the action for ϕ that we have used above. Thus we will have

$$< X(x)X(0) >= -\alpha' \ln \frac{|x|}{a} \tag{38}$$

or in units where $\alpha' = 1$

$$< X(x)X(0) >= -\ln \frac{|x|}{a} \tag{39}$$

4 Complex coordinates

We will use the complex coordinates

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2 \tag{40}$$

Then

$$z\bar{z} = |x|^2 \tag{41}$$

and

$$< \phi(z)\phi(0) >= -\frac{1}{4\pi}\ln(z\bar{z}) \tag{42}$$

5 The operator $\partial\phi$

We can compute different derivatives of ϕ. In complex coordinates we get

$$\partial_z = \partial_{x_1}\frac{\partial x_1}{\partial z} = \partial_{x_2}\frac{\partial x_2}{\partial z} \tag{43}$$

But

$$x_1 = \frac{1}{2}(z+\bar{z}), \quad x_2 = \frac{1}{2i}(z-\bar{z}) \tag{44}$$

so we get

$$\partial_z = \frac{1}{2}[\partial_{x_1} - i\partial_{x_2}] \tag{45}$$

Let us compute

$$< \partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) >= -\frac{1}{4\pi}\partial_{z_1}\partial_{z_2}\ln[(z_1-z_2)(\bar{z}_1-\bar{z}_2)] = \frac{1}{4\pi}\partial_{z_1}^2\ln[(z_1-z_2)(\bar{z}_1-\bar{z}_2)] \tag{46}$$

As long as $z_1 - z_2 \neq 0$ we can write

$$< \partial_z\phi(z_1)\partial_z\phi(z_2) >= \frac{1}{4\pi}\partial_{z_1}^2\ln[(z_1-z_2)] = -\frac{1}{4\pi}\frac{1}{(z_1-z_2)^2} \tag{47}$$

Thus we see that the scaling dimension of $\partial_z\phi$ is 1, as expected.

In general we just write $\partial_z\phi(z_1)$ or $\partial\phi(z_1)$ instead of the full expression $\partial_{z_1}\phi(z_1)$. It is assumed that the derivative is with respect to the argument of the field on which the derivative acts.

6 The operator $e^{i\alpha\phi(x)}$

We will be interested in the operator

$$e^{i\alpha\phi(x)} \tag{48}$$

Such operators will appear naturally in string theory. If we think of ϕ a an operator with scaling dimension zero, then it would appear that this exponential should also have dimension zero. But we have seen that ϕ does not have a well defined dimension, and we will find that the exponential has a well defined scaling dimension that is *not* zero.

Let us study the correlator

$$< e^{i\alpha\phi(x)} e^{-i\alpha\phi(0)} > \tag{49}$$

Let us expand each exponential in a power series. We get

$$< \sum_n \frac{1}{n!}(i\alpha)^n[\phi(x)]^n \sum_m \frac{1}{m!}(-i\alpha)^m[\phi(0)]^m > \tag{50}$$

By Wick's theorem, we should contract away all the scalar fields in pairs. Let us assume that the operators are normal ordered. So we can contract the operators at one point only with operators at the other point. To get a complete contraction, we see that we can only have terms with $n = m$. We see that there are $n!$ ways to make this contraction, and each gives a contribution

$$[-\frac{1}{2\pi}\ln|x|]^n \tag{51}$$

Thus we find

$$< e^{i\alpha\phi(x)} e^{-i\alpha\phi(0)} >= \sum_n \frac{1}{n!}\alpha^{2n}[-\frac{1}{2\pi}\ln|x|]^n = e^{-\alpha^2\frac{1}{2\pi}\ln|x|} = |x|^{-\frac{\alpha^2}{2\pi}} \tag{52}$$

Thus the dimension of the operator $e^{i\alpha\phi}$ is

$$\Delta = \frac{\alpha^2}{4\pi} \tag{53}$$

7 The operators in string theory

In string theory we will look at operators like

$$e^{ikX} = e^{i\sqrt{2\pi\alpha'}k\phi} \tag{54}$$

The dimension of this operator will be

$$\Delta = \frac{2\pi\alpha' k^2}{4\pi} = \frac{\alpha' k^2}{2} \tag{55}$$

In units where $\alpha' = 1$ we have

$$\Delta = \frac{k^2}{2} \tag{56}$$

This is the total scaling dimension, and if we look at the holomorphic and antiholomorphic parts separately then we will find

$$(\frac{k^2}{4}, \frac{k^2}{4}) \tag{57}$$

8 Vertex operators

The string worldsheet is 2-dimensional. We have to do a path integral over fields X^μ on this surface. For the moment let us ignore the constraints. Recall that the action was conformally invariant, and if we insert any operators to make a correlation function then we want this correlation function to be conformally invariant as well. But where shall we insert the operator? The only covariant way is to insert it at a point z and then integrate over z. The integration measure will be

$$d^2z = dzd\bar{z} \tag{58}$$

This has scaling dimension L^2 or $(mass)^{-2}$. Thus the inserted operator should have mass dimension 2. More precisely, the integral measure has mass dimensions $(-1,-1)$ in $z, \bar{z}$, and we need the inserted operator to have mass dimension $(1,1)$ in $z, \bar{z}$. Let us see how this can be achieved.

The simplest operators are just the exponential functions e^{ikX}. To get $\Delta = 1$ for z we need

$$\frac{k^2}{4} = 1 \tag{59}$$

Since this is positive, we have a spacelike momentum k. This vertex operator therefore corresponds to a tachyon. We see that

$$k^2 = 4 \tag{60}$$

just as we had found from our analysis of tachyon state from the analysis of string states.

The next operator that we can make can have the form

$$\partial_z X^\mu e^{ikX} \tag{61}$$

Now the operator ∂_z already supplies the mass dimension 1, so we need

$$k^2 = 0 \tag{62}$$

so we have a massless particle. The same holds for the $\bar{z}$ side, so the overall operator has the form

$$\partial_z X^\mu \bar{\partial} X^\nu e^{ikX} \tag{63}$$

and we see that we have the right indices to describe a graviton.

9 The transfer matrix for the scalar field

Let us now take the free scalar field in $0+1$ dimensions. Let the τ direction be on a lattice with spacing Δ as above. The action from one slice between two lattice points is

$$S = \frac{1}{2}\dot{\phi}^2 d\tau = \frac{1}{2}(\frac{\phi_{i+1}-\phi_i}{\Delta})^2\Delta = \frac{1}{2}\frac{(\phi_{i+1}-\phi_i)^2}{\Delta} \tag{64}$$

The state at site i can be any value of ϕ. Each if these classical possibilities will be denoted by

$$|\phi\rangle \tag{65}$$

The state at site $i+1$ will be similarly given by a value

$$|\phi'\rangle \tag{66}$$

The transfer matrix between these two possibilities will be

$$e^{-\frac{1}{2\Delta}(\phi'-\phi)^2} \tag{67}$$

This is not diagonal, since ϕ need not equal ϕ'. But we can make formal linear combinations of ϕ

$$|k\rangle = \frac{1}{\sqrt{2\pi}}\int d\phi e^{ik\phi}|\phi\rangle \tag{68}$$

Between two such vectors, $|k\rangle$ at i and $|k'\rangle$ at $i+1$ we will have for the transfer matrix

$$M_{kk'} = \int d\phi \int d\phi' e^{-ik'\phi'} e^{ik\phi} e^{-\frac{1}{2\Delta}(\phi'-\phi)^2} \tag{69}$$

But we have

$$e^{-ik'\phi'}e^{ik\phi} = e^{\frac{i(k-k')(\phi+\phi')}{2}} e^{\frac{i(k+k')(\phi-\phi')}{2}} \tag{70}$$

and

$$d\phi d\phi' = \frac{1}{2}d(\phi-\phi')d(\phi+\phi') \tag{71}$$

Thus we have

$$M_{kk'} = \frac{1}{2\pi}\frac{1}{2}\int d(\phi-\phi')\int d(\phi+\phi')e^{\frac{i(k-k')(\phi+\phi')}{2}} e^{\frac{i(k+k')(\phi-\phi')}{2}} e^{-\frac{1}{2\Delta}(\phi'-\phi)^2} \tag{72}$$

We have

$$\int d(\phi+\phi')e^{\frac{i(k-k')(\phi+\phi')}{2}} = 2\pi\delta(\frac{k-k'}{2}) = 4\pi\delta(k-k') \tag{73}$$

$$\int d(\phi-\phi')e^{-\frac{1}{2\Delta}(\phi'-\phi)^2} e^{\frac{i(k+k')(\phi-\phi')}{2}} = \sqrt{2\pi}e^{-\frac{\Delta(k+k')^2}{8}} \tag{74}$$

The delta function implies that $k=k'$ so we can write the above result as

$$\sqrt{2\pi}e^{-\frac{\Delta}{2}k^2} \tag{75}$$

So overall we find

$$M_{kk'} = (2\pi)^{\frac{1}{2}}\delta(k-k')e^{-\frac{\Delta}{2}k^2} \tag{76}$$

Now the transfer matrix is diagonal. We also see that

$$k = \frac{1}{i}\frac{\delta}{\delta\phi} = \pi_\phi \tag{77}$$

Thus

$$k^2 = -\frac{\delta^2}{\delta\phi^2} = \phi_\phi^2 \tag{78}$$

We see that

$$M_{kk'} = (2\pi)^{\frac{1}{2}}\delta(k-k')e^{-\frac{\Delta}{2}\phi_\phi^2} \tag{79}$$

This gives the Hamiltonian evolution in the quantized theory, with the continuum limit

$$-\frac{\Delta}{2}\phi_\phi^2 \to \frac{1}{2}d\tau\pi_\phi^2 = d\tau H \tag{80}$$

with

$$H = \frac{1}{2}\pi_\phi^2 \tag{81}$$

10 Correlator between exponentials and ∂X

We use the field X with correlation function

$$< X(z)X(z') = -\ln\frac{|z|}{a} = \frac{1}{2}[\ln\frac{z}{a} + \ln\frac{\bar{z}}{a}] \tag{82}$$

where we have set $\alpha' = 1$.

Let us compute the correlation function

$$< \partial X(z)e^{ikX}(z') > \tag{83}$$

We expand the exponential in a power series, getting

$$< \partial X(z)\sum_{m=0}^{\infty}\frac{(ik)^m}{m!}[X(z')]^m > \tag{84}$$

We have

$$< \partial X(z)X(z') >= -\partial_z\frac{1}{2}\ln z = -\frac{1}{2z} \tag{85}$$

In the Wick contractions, the ∂X can contract with any of the m terms $X(z')$, so we get

$$-\frac{1}{2z}(ik)\sum_{m=0}^{\infty}\frac{(ik)^{m-1}}{(m-1)!}[X(z')]^{m-1} = -\frac{1}{2z}e^{ikX}(z') \tag{86}$$

Thus

$$< \partial X(z)e^{ikX}(z') >= -\frac{1}{2z}e^{ikX}(z') \tag{87}$$

11 The operator product expansion

We have

$$< \partial X(z)\partial X(z') >= -\frac{1}{2}\frac{1}{(z-z')^2} \tag{88}$$

This is what we get is we want to compute a correlator of just $\partial X(z)$ and $\partial X(z')$. But suppose there are other operators in the correlator

$$< X(z)X(z')O_1(z_1)\ldots O_n(z_n) > \tag{89}$$

What can we do with the two operators ∂X now? In general, nothing, since we cannot just contract them by a Wick contraction and remove them. This is because while there is certainly a term with such a contraction, there will also be terms where $\partial X(z)$ contracts with an X in some operator O_k and $\partial X(z')$ also contracts with an X in some $O_{k'}$. So it would seem that we cannot simplify this expression in general without knowing something more about the operators O_k.

But suppose that z is close to z', in the sense that

$$|z - z'| \ll |z - z_k| \tag{90}$$

so the two points with the operators ∂X are much closer to each other than to any of the operators O_k. In this case we can replaces them by an *operator product expansion.* This is done as follows.

First we look at the term where the two ∂X operators do Wick contract. This removes them as operators from the correlator, but gives a contribution

$$-\frac{1}{2}\frac{1}{(z-z')^2} \tag{91}$$

If $z - z'$ is small, this will be larger than other contributions where do do not have this Wick contraction, so in a series expansion it is reasonable that this be the first term. What we do is write

$$\partial X(z)\partial X(z') = -\frac{1}{2}\frac{1}{(z-z')^2}I(z') + \ldots \tag{92}$$

where I is the identity operator. Inserting the identity operator at any point does not make any change to the correlator. So thus far in the expansion we have

$$\begin{aligned} < X(z)X(z')O_1(z_1)\ldots O_n(z_n) > &= < [-\frac{1}{2}\frac{1}{(z-z')^2}I(z') + \ldots]O_1(z_1)\ldots O_n(z_n) > \\ &= -\frac{1}{2}\frac{1}{(z-z')^2} < I(z')O_1(z_1)\ldots O_n(z_n) > + \ldots \\ &= -\frac{1}{2}\frac{1}{(z-z')^2} < O_1(z_1)\ldots O_n(z_n) > + \ldots \end{aligned} \tag{93}$$

Now we have to look at terms where we do not Wick contract these two ∂X operators with each other. We do not know much about the operators O_k, so we cannot actually say what other contractions are possible. So we have to leave the two ∂X operators as they are, but we do have to note that they should not Wick contract between themselves, since we have already taken that contribution into account. We can define an operator

$$:\partial X \partial X: \tag{94}$$

where the normal ordering symbol means that we should not contract these two operators among themselves. But if we are to think of this as a local operator then we have to ask at what point it sits. Note that one ∂X was at z and one at z'. Since $z - z'$ is small, we can at leading order put both ∂X operators at z'. This gives

$$\partial X(z)\partial X(z') = -\frac{1}{2}\frac{1}{(z-z')^2}I(z') + :\partial X \partial : X(z') + \ldots \tag{95}$$

Let us see in more detail what is the meaning of putting both operators ∂X at the same point z'. These operators were going to be used in Wick contractions with X operators in the O_k, at which points we would just get c-number functions. One operator ∂X is already at z', so we make no further changes to its position. The other operator can be written as

$$\partial X(z) = \partial X(z') + (z-z')\partial^2 X(z') + \frac{(z-z')^2}{2}\partial^3 X(z') + \ldots \tag{96}$$

It can be easily seen that if we insert this expansion in the correlator and use it to compute the Wick contractions of $\partial X(z)$ then we will get the correct result. Thus our complete operator product expansion becomes

$$\partial X(z)\partial X(z') = -\frac{1}{2}\frac{1}{(z-z')^2}I(z')+(z-z') : \partial X\partial X : (z')+ : \partial^2 X\partial X : (z')+\frac{1}{2}(z-z')^2 : \partial^3 X\partial X : (z')+\ldots \tag{97}$$

We have to put the normal ordering symbol on all terms after he first to tell us that we should not Wick contract these operators. Since $z-z'$ is small, the terms have coefficients that are decreasing. Thus the first few terms on the RHS would furnish a good approximation if inserted in the correlator in place of the operators on the LHS. This i called the operator product expansion. In general it has the form

$$O(z,\bar{z})O'(z',\bar{z}') = (z-z')^{\alpha_1}(\bar{z}-\bar{z}')^{\bar{\alpha}_1}O_1(z',\bar{z}') + (z-z')^{\alpha_2}(\bar{z}-\bar{z}')^{\bar{\alpha}_2}O_2 + \ldots.. \tag{98}$$

where $\alpha_1 < \alpha_2 < \alpha_3$ etc..