1 Light cone coordinates on the world sheet

On the world sheet we have used the coordinates τ, σ. We will see however that the physics is simpler in light cone coordinates

$$\xi^+ = \tau + \sigma, \quad \xi^- = \tau - \sigma \tag{1}$$

Then we have

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\xi_+} + \frac{\partial}{\partial\xi_-} \equiv \partial_+ + \partial_- \tag{2}$$

$$\frac{\partial}{\partial\sigma} = \frac{\partial}{\partial\xi_+} - \frac{\partial}{\partial\xi_-} \equiv \partial_+ - \partial_- \tag{3}$$

The wave equation

$$[\partial_\tau^2 - \partial_\sigma^2]X^\mu = 0 \tag{4}$$

becomes

$$\partial_+\partial_- x^\mu = 0 \tag{5}$$

The metric on the world sheet is

$$ds^2 = g_{++}(d\xi^+)^2 + g_{--}(d\xi^-)^2 + 2g_{+-}(d\xi_+)(d\xi_-) \tag{6}$$

If the metric has the conformal form that we have chosen, then we have

$$g_{++} = g_{--} = 0, \quad g_{+-} = -\frac{1}{2}e^{2\rho} \tag{7}$$

2 The equation of motion for g_{ab}

Recall that we have taken the action

$$S = -T\int d^2\xi\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{8}$$

Consider the condition

$$\frac{\delta S}{\delta g^{ab}} = 0 \tag{9}$$

For the world sheet action above, this means that the stress-energy tensor of the world sheet theory vanishes

$$T_{ab}(\xi) \equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}(\xi)} = 0 \tag{10}$$

Working out the variation of the action we find

$$T_{ab} = T[\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}g_{ab}\partial_c X^\mu \partial^c X_\mu] = 0 \tag{11}$$

Note that the trace of this stress tensor vanishes

$$T^c{}_c = 0 \tag{12}$$

There were three independent components of the stress tensor since $T_{ab} = T_{ba}$, and with the automatic vanishing of the trace we find that we are left with two vanishing conditions. Let us examine these conditions in the coordinate frame where the metric is conformal to η_{ab}. Then we find

$$T_{10} = T_{01} = T\dot{X} \cdot X' \tag{13}$$

$$T_{00} = T_{11} = T\frac{1}{2}[\dot{X} \cdot \dot{X} + X' \cdot X'] \tag{14}$$

In the coordinates $\xi^{\pm}$, we get

$$T_{++} = \frac{1}{2}[T_{00} + T_{11} + 2T_{01}] = \frac{1}{2}(\dot{X} + X') \cdot (\dot{X} + X') = X_{,+} \cdot X_{,+} = 0 \tag{15}$$

$$T_{--} = \frac{1}{2}[T_{00} + T_{11} - 2T_{01}] = \frac{1}{2}(\dot{X} - X') \cdot (\dot{X} - X') = X_{,-} \cdot X_{,-} = 0 \tag{16}$$

3 Finding the momentum of the string

We will soon see that we can write simple solution to the $X^\mu(\tau, \sigma)$. In the general solution for X^μ, we will encounter a form

$$X^\mu(\tau, \sigma) = x^\mu + a^\mu \tau + \ldots \tag{17}$$

One would think intuitively that a^μ would give the momentum P^μ carried by the string in spacetime. But how do we show this, and what is the constant relating a^μ to p^μ?

Let us start with the case of the point particle. We had the action

$$S = -m \int d\tau \sqrt{-\frac{\partial X^\mu}{\partial \tau} \frac{X_\mu}{\partial \tau}} \tag{18}$$

We can of course regard this action as as describing fields X^μ over the 1-dimensional base space τ. Since the action has a translation symmetry

$$X^\mu \to X^\mu + \epsilon^\mu \tag{19}$$

we expect to get conserved Noether currents for these symmetries. The conserved current for translations of X^μ is

$$J_\tau = -\frac{m}{\sqrt{-\frac{\partial X^\nu}{\partial \tau} \frac{\partial X_\nu}{\partial \tau}}} \frac{\partial X_\mu}{\partial \tau} \tag{20}$$

It is tempting to say that this should give the momentum p_μ of the particle in spacetime. In fact the answer happens to be correct, but the reasoning has a flaw. If we want the momentum of the quantum in spacetime, then we cannot just look at the Noether current on the *worldline*; there is no relation between these two notions in general. It is possible to make simple examples where the spacetime conserved quantity and the conserved quantity obtained from the worldsheet action are different, but for the moment let us ask why they are often the *same*. This happens because there are not too many conserved quantities that one can make with the given variables, so if we have found a conserved quantity from the worldsheet action, it might well be the conserved quantity from the spacetime point of view.

In any case, how do we find the actual spacetime p^μ? The first principles definition of p^μ is as the spatial integral of $T_{0\mu}$

$$P_\mu = \int d^{D-1}X^i T_{0\mu} = \int d^{D-1}X^i \frac{2}{\sqrt{-G}}\frac{\delta S}{\delta G^{0\mu}} \tag{21}$$

where $G_{\mu\nu}$ is the metric of *spacetime*. To apply this definition to our problem, we have to first write the action for the point particle in a spacetime with general metric $G_{\mu\nu}$. We have

$$S = -m\int d\tau\sqrt{-\frac{dX_\mu}{\partial\tau}\frac{dX_\nu}{\partial\tau}}G^{\mu\nu} \tag{22}$$

We will rewrite this in a way that makes the action S a distribution over the D-dimensional spacetime

$$S = -m\int D^D X\int d\tau\sqrt{-\frac{d\bar{X}_\mu(\tau)}{d\tau}\frac{d\bar{X}_\nu(\tau)}{d\tau}}G^{\mu\nu}\delta^D(X^\mu - \bar{X}^\mu(\tau)) \tag{23}$$

Here $\bar{X}^\mu(\tau)$ is the trajectory of the particle, and the delta function just says that the nonvanishing contribution to S comes only from points on this worldline. We will make a variation with respect to $G^{\mu\nu}$ and then set $G^{\mu\nu} = \eta^{\mu\nu}$. We find

$$T_{\mu\nu} = \frac{2}{\sqrt{-G}}\frac{\delta S}{\delta G^{\mu\nu}} = m\int d\tau\delta^D(X^\mu - \bar{X}^\mu(\tau))\frac{\frac{dX_\mu}{d\tau}\frac{dX_\nu}{d\tau}}{\sqrt{-\frac{d\bar{X}_\mu(\tau)}{d\tau}\frac{d\bar{X}_\nu(\tau)}{d\tau}}} \tag{24}$$

Thus

$$p_\mu = \int d^{D-1}X^i T_{0\mu} = m\int d\tau\delta(X^0 - \bar{X}^0(\tau))\frac{\frac{dX_0}{d\tau}\frac{dX_\mu}{d\tau}}{\sqrt{-\frac{d\bar{X}^\lambda(\tau)}{d\tau}\frac{d\bar{X}_\lambda(\tau)}{d\tau}}} \tag{25}$$

We can now use

$$\int d\tau\delta(X^0 - \bar{X}^0(\tau)) = [\frac{dX^o}{d\tau}]^{-1} \tag{26}$$

to get

$$p_\mu = m\frac{d\bar{X}_\mu}{d\tau}\frac{1}{\sqrt{-\frac{d\bar{X}^\lambda(\tau)}{d\tau}\frac{d\bar{X}_\lambda(\tau)}{d\tau}}} \tag{27}$$

so we see that in this case we do get the result obtained from the world sheet calculation.

Let us now do the same for the string. We take the action

$$S = -T\int d\tau d\sigma\sqrt{-g}\frac{1}{2}\frac{\partial X_\mu}{\partial\xi^c}\frac{\partial X_\nu}{\partial\xi^d}g_{cd}G^{\mu\nu} \tag{28}$$

As before we write this as

$$S = -T\int d^D X\int d\tau d\sigma\delta^D[X^\mu - \bar{X}^\mu(\tau,\sigma)]\sqrt{-g}\frac{1}{2}\frac{\partial\bar{X}_\mu}{\partial\xi^c}\frac{\partial\bar{X}_\nu}{\partial\xi^d}g_{cd}G^{\mu\nu} \tag{29}$$

Let the worldsheet be put in the conformal gauge. Then we find

$$p_\mu = \int d^{D-1}X^- T_{0\mu} = T[\frac{\partial\bar{X}_0}{\partial\tau}\frac{\partial\bar{X}_\nu}{\partial\tau} - \frac{\partial\bar{X}_0}{\partial\sigma}\frac{\partial\bar{X}_\nu}{\partial\sigma}]\delta(X^0 - \bar{X}^0(\tau,\sigma))d\tau d\sigma \tag{30}$$

In the first piece we can solve the delta function as before, getting

$$p_\mu = T \int d\sigma \frac{\partial \bar{X}_\nu}{\partial \tau} \tag{31}$$

The other piece vanishes because the σ integral is periodic. Thus we have an expression for p_μ that is similar to the one in the point particle case, and it is again one that could have been guessed (though not rigorously derived) from the worldsheet action and its Noether currents.

4 The mode expansion: zero modes

Let us write the harmonic functions X^μ as

$$X^\mu(\tau, \sigma) = x^\mu + a^\mu \tau + b^\mu \sigma + \ldots \tag{32}$$

where the terms that are not written involve the sinusoidal oscillations of the string. We will focus on a^μ, b^μ in this section.

Note that x^μ gives the center of mass position of the string. We can guess that a^μ will be related to the center of mass momentum carried by the string. Let us find the precise connection between a^μ and p^μ.

We consider the closed string. Let us take the range of σ to be

$$0 \le \sigma < 2\pi \tag{33}$$

From

$$p_\mu = T \int d\sigma \frac{\partial X_\nu}{\partial \tau} \tag{34}$$

we conclude that

$$p_\mu = 2\pi T a_\mu \tag{35}$$

so that

$$a_\mu = \frac{1}{2\pi T} p_\mu \tag{36}$$

We will write

$$T = \frac{1}{2\pi\alpha'} \tag{37}$$

so that α' will have units of L^2. Then we get

$$a_\mu = \alpha' p_\mu \tag{38}$$

and our string mode expansion looks like

$$X^\mu(\tau, \sigma) = x^\mu + \alpha' p^\mu \tau + b^\mu \sigma + \ldots \tag{39}$$

Now let us consider b^μ. The string is a closed loop, so we might expect that $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$. This would imply that $b^\mu = 0$. But recall that some of the directions of spacetime can be closed circles, and the string can wrap around these closed circles. Let the compact directions form a torus, which is specified by giving a lattice of identifications on R^n, where n is the number of compact directions. Let the vectors defining the sides of the torus be given by

$$L^\mu_{(\alpha)}, \quad \alpha = 1, \ldots n \tag{40}$$

Then we must have

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) + \sum_{(\alpha)} n_{(\alpha)} L^\mu_{(\alpha)} \tag{41}$$

From the mode expansion we thus see that

$$2\pi b^\mu = \sum_{(\alpha)} n_{(\alpha)} L^\mu_{(\alpha)} \tag{42}$$

or

$$b^\mu = \frac{1}{2\pi} \sum_{(\alpha)} n_{(\alpha)} L^\mu_{(\alpha)} \tag{43}$$

Let us now return to the mode expansion. We have noted that the most general solution to the wave equation can be written as a function of $\xi^+ = \tau + \sigma$ plus a function of $\xi^- = \tau - \sigma$. Thus let us write our mode expansion in this form

$$\begin{aligned} X^\mu(\tau, \sigma) &= [\frac{1}{2}x^\mu + \frac{1}{2}(a^\mu + b^\mu)(\tau + \sigma) + \ldots] \\ &\quad +[\frac{1}{2}x^\mu + \frac{1}{2}(a^\mu - b^\mu)(\tau - \sigma) + \ldots] \\ &\equiv [\frac{1}{2}x^\mu + \alpha' p^\mu_L(\tau + \sigma) + \ldots] \\ &\quad +[\frac{1}{2}x^\mu + \alpha' p^\mu_R(\tau - \sigma) + \ldots] \end{aligned} \tag{44}$$

Thus we get

$$p^\mu_L = \frac{1}{2}[p^\mu + \frac{1}{2\pi\alpha'} n_{(\alpha)} L^\mu_{(\alpha)}] = \frac{1}{2}[p^\mu + T n_{(\alpha)} L^\mu_{(\alpha)}] \tag{45}$$

$$p^\mu_R = \frac{1}{2}[p^\mu - \frac{1}{2\pi\alpha'} n_{(\alpha)} L^\mu_{(\alpha)}] = \frac{1}{2}[p^\mu - T n_{(\alpha)} L^\mu_{(\alpha)}] \tag{46}$$

So far we have been looking at the classical theory, but if we quantize the string then the momentum in the compact directions will be quantized. From single-valuedness of the wavefunction we will need to have

$$e^{ip_\mu X^\mu} = e^{ip_\mu\left(X^\mu + \sum_{(\alpha)} n_{(\alpha)} L^\mu_{(\alpha)}\right)} \tag{47}$$

which implies that

$$p_\mu L^\mu_{(\alpha)} = 2\pi m_{(\alpha)} \tag{48}$$

for integral $m_{(\alpha)}$.

From the above we note the following fact

$$p_L^2 - p_R^2 = \frac{1}{2\pi\alpha'} \sum_{(\alpha)} n_{(\alpha)} p_\mu L^\mu_{(\alpha)} = \frac{1}{\alpha'} \sum_{(\alpha)} n_{(\alpha)} m_{(\alpha)} \tag{49}$$

Thus $\alpha'(p_L^2 - p_R^2)$ is an integer. This fact will be of importance when we study black holes.

5 The full mode expansion

Now let us write the full expansion of X^μ

$$X^\mu(\tau,\sigma) = [\frac{1}{2}x^\mu + \alpha' p_L^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau+\sigma)}] + [\frac{1}{2}x^\mu + \alpha' p_R^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-in(\tau-\sigma)}] \tag{50}$$

We have just written added all the nonzero harmonics for the left and right movers. At the present classical level, observe that reality of the X^μ implies

$$\alpha_n^\mu = \left(\alpha_{-n}^\mu\right)^* \tag{51}$$

At the quantum level we will have

$$\hat{\alpha}_n^\mu = \left(\hat{\alpha}_{-n}^\mu\right)^\dagger \tag{52}$$

The reason for writing the coefficients in the above form will be clear when we compute the commutators of the α_n^μ, since they will look simple

$$[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{53}$$

Let us verify this commutation relation. The action is

$$S = -T\int d^2\xi \frac{1}{2}\partial_c X^\mu \partial^c X_\mu \tag{54}$$

The basic variables are X^μ, and the canonically conjugate momenta in this 2-dimensional action are

$$P_\mu = \frac{\partial L}{\partial_\tau X^\mu} = T\partial_\tau X_\mu \tag{55}$$

With the above mode expansion we get

$$P_\mu = T\left[\alpha' p_\mu + \sqrt{\frac{\alpha'}{2}}\alpha_{\mu n}e^{-in(\tau-\sigma)} + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_{\mu n}e^{-in(\tau+\sigma)}\right] \tag{56}$$

In the commutator $[X^\mu, P_\nu]$ let us first look at the contribution of the left oscillators. We set the times equal $\tau = \tau' = 0$ and get

$$[X^\mu(\sigma), P_\nu(\sigma')] \to \delta_\nu^\mu T\frac{\alpha'}{2}i[\frac{\alpha_m}{m}, \alpha_n]e^{im\sigma}e^{in\sigma'} \tag{57}$$

Using the above commutation relation between oscillators, this gives

$$[X^\mu(\sigma), P_\nu(\sigma')] \to \delta_\nu^\mu \frac{1}{4\pi}i\sum_{m\neq 0}e^{im(\sigma-\sigma')} \tag{58}$$

The zero modes contribute

$$[x^\mu, T\alpha' p_\nu] = i\delta_\nu^\mu\frac{1}{2\pi} \tag{59}$$

Putting half of this contribution towards the left movers and half towards the right movers, we find that the left movers give

$$[X^\mu(\sigma), P_\nu(\sigma')] \to \delta^\mu_\nu \frac{1}{4\pi} i \sum_n e^{in(\sigma-\sigma')} = i\delta^\mu_\nu \frac{1}{2}\delta(\sigma-\sigma') \tag{60}$$

Adding in the contribution of the right movers, we get

$$[X^\mu(\sigma), P_\nu(\sigma')] = \delta^\mu_\nu i\delta(\sigma-\sigma') \tag{61}$$

as desired.

6 The constraints

Now recall the constraints

$$\frac{\partial X^\mu}{\partial \xi^+}\frac{\partial X_\mu}{\partial \xi^+} = 0, \qquad \frac{\partial X^\mu}{\partial \xi^-}\frac{\partial X_\mu}{\partial \xi^-} = 0 \tag{62}$$

We have

$$\frac{\partial X^\mu}{\partial \xi^+} = \alpha' p^\mu_L + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha^\mu_n e^{-in\xi^+} \tag{63}$$

We write

$$\alpha' p^\mu_L = \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0, \qquad \Rightarrow \qquad \alpha^\mu_0 = \sqrt{2\alpha'} p^\mu_L \tag{64}$$

Then

$$\frac{\partial X^\mu}{\partial \xi^+} = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha^\mu_n e^{-in\xi^+} \tag{65}$$

where now the term $n = 0$ is included in the sum. Then

$$\frac{\partial X^\mu}{\partial \xi^+}\frac{\partial X_\mu}{\partial \xi^+} = \frac{\alpha'}{2}\left(\sum_m \alpha^\mu_m e^{-im\xi^+}\right)\left(\sum_n \alpha^\mu_n e^{-in\xi^+}\right) \tag{66}$$

This function should vanish for all τ, σ. Thus the coefficient of all fourier modes should be zero. We get

$$\eta_{\mu\nu}\alpha^\mu_m {\alpha^\nu}_{p-m} = 0, \quad \text{for all } p \tag{67}$$

where from now on we will assume the summation convention on indices like m which are repeated.

Consider the above constraint for $p = 0$. We get

$$2\alpha' p^2_L + \eta_{\mu\nu}\alpha^\mu_m {\alpha^\nu}_{-m} = 0 \tag{68}$$

We have let n of the directions be compact. Let p^μ_{nc} be the momentum of the string in the noncompact directions. The mass of the state as seen from the viewpoint of these noncompact directions is

$$m^2 = -p^\mu_{nc} p_{nc\mu} \tag{69}$$

Thus we should separate the compact and noncompact contributions to p_L^2. We will assume that the compact directions are all completely orthogonal to the noncompact ones. Let $p^\mu_{c,L}$ be the part of p^μ_L in the compact directions. Then we have

$$m^2 = 4p^\mu_{c,L}p_{c,L\mu} + \frac{2}{\alpha'}\alpha^\mu_m\alpha^\nu{}_{-m} \tag{70}$$

Repeating the same computations with the left movers we can get a similar relation

$$m^2 = 4p^\mu_{c,R}p_{c,R\mu} + \frac{2}{\alpha'}\tilde{\alpha}^\mu_m\tilde{\alpha}^\nu{}_{-m} \tag{71}$$

Subtracting these two expressions for m^2 we find

$$p^\mu_{c,L}p_{c,L\mu} - p^\mu_{c,R}p_{c,R\mu} = -\frac{1}{2\alpha'}\left[\alpha^\mu_m\alpha^\nu{}_{-m} - \tilde{\alpha}^\mu_m\tilde{\alpha}^\nu{}_{-m}\right] \tag{72}$$

But for the noncompact directions $p^\mu_L = p^\mu_R = \frac{1}{2}p^\mu$. Thus we have

$$p^\mu_{c,L}p_{c,L\mu} - p^\mu_{c,R}p_{c,R\mu} = p_L^2 - p_R^2 \tag{73}$$

and we find

$$p_L^2 - p_R^2 = -\frac{1}{2\alpha'}\left[\alpha^\mu_m\alpha^\nu{}_{-m} - \tilde{\alpha}^\mu_m\tilde{\alpha}^\nu{}_{-m}\right] \tag{74}$$

Recalling (49) we see that

$$\alpha^\mu_m\alpha^\nu{}_{-m} - \tilde{\alpha}^\mu_m\tilde{\alpha}^\nu{}_{-m} = 2N \tag{75}$$

where N is an integer. This expression will be important in the computation of black hole entropy.

7 The Casimir effect

The zero point energies for a field mode give a contribution

$$\frac{1}{L}\frac{1}{2}\left[1+2+\ldots\right] \tag{76}$$

Thus let us compute the sum

$$S = 1+2+\ldots \tag{77}$$

Formally, this sum looks infinite. But physically, what we want is the energy for these discrete modes, minus the energy from the same region of space if the modes were continuous. To compute the difference of these two infinite quantities, we must first regularize each of them. To do this, choose any function $f(x)$ which is unity for small x, but at very large x (say $x \gg Q$) goes down smoothly to zero. Then we compute

$$\sum_0^\infty nf(n) - \int_0^\infty xf(x)dx$$

At the end we take $Q \to \infty$, and this should give us the required difference in energies.

To see this computation pictorially, let us plot the function $xf(x)$. Note that the integral under this curve gives the contribution of continuous vacuum modes, while if we just sum this function over the integer values $x = n$ then we get the discrete sum. We can draw a 'bar graph' where we draw a rectangle with height n over the interval from $n < x < n+1$. Then the discrete sum is the area under this bar graph. The difference in areas under the bar graph and under the curve will be the difference that we are seeking. Remarkably, this will be a finite number, and in the limit of large Q, will be independent of the choice of function f.

We will call the discrete sum S, and the continuous integral I.

Look at the block from $\bar{n}$ to $\bar{n}+1$. We get

$$Sum = S = nf(n)$$

$$I = \int_n^{n+1} xf(x)dx = \int_0^1 dy[nf(n)] + [nf'(n) + f(n)]y + [2f'(n) + nf''(n)]\frac{y^2}{2}]$$

$$= [nf(n)] + [nf'(n) + f(n)]\frac{1}{2} + [2f'(n) + nf''(n)]\frac{1}{6}]$$

So we have

$$S - I = -\sum \frac{1}{2}[nf'(n) + f(n)] - \frac{1}{6}[2f'(n) + nf''(n)]$$

Note that we have large envelope for f, and we can take each of its derivatives to be $O(1/Q)$, where Q is the size of this envelope. Thus we see that the first set of terms is larger than the second.

Let us write

$$\tilde{S} = \sum[nf'(n) + f(n)]$$

Then we can define

$$\tilde{I} = \int dx \left(xf'(x) + f(x)\right)$$

Let us compute this in the sam way as above, approximating it by an integral. We have

$$\tilde{I} = \sum[nf'(n) + f(n)]y + [2f'(n) + nf''(n)]\frac{y^2}{2}$$

$$= \sum[nf'(n) + f(n)] + [2f'(n) + nf''(n)]\frac{1}{2}$$

$$\tilde{S} - \tilde{I} = -\sum[2f'(n) + nf''(n)]\frac{1}{2}$$

Note that

$$\tilde{I} = xf(x)|_0^\infty - \int f(x) + \int f(x) = 0$$

We thus have

$$S - I = \sum[f'(n)\frac{1}{2} + nf''(n)\frac{1}{4} - f'(n)\frac{1}{3} - nf''(n)\frac{1}{6} = \sum \frac{1}{6}f'(n) + \frac{1}{12}nf''(n)$$

Let us now again approximate this by an integral; this time we will find that the difference between the sum and the integral is ignorable at large Q. Then we get

$$S - I = \int dx[\frac{1}{6}f'(x) + \frac{1}{12}xf''(x)]$$

$$
\begin{aligned}
&= \frac{1}{6}(-1) + \frac{1}{12}[f'(x)x| - \int f'(x)] \\
&= -\frac{1}{6} + \frac{1}{12} \\
&= -\frac{1}{12}
\end{aligned}
$$

where we have used that $f(0) = 1, f'(0) = 0$.