1 Poisson brackets

Hamilton's equations are

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}\\ \dot{p} &= -\frac{\partial H}{\partial q}\end{aligned} \tag{1}$$

These equations can be written in a simple form if we define the Poisson bracket of two functions f, g of the variables q, p

$$\{f(q,p), g(q,p)\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{2}$$

Then we have

$$\begin{aligned}\dot{q} &= \{q, H(q,p)\}\\ \dot{p} &= \{p, H(q,p)\}\end{aligned} \tag{3}$$

so the equations look completely symmetrical; now there is no sign difference between the two equations. More generally, for any function of q, p

$$\frac{d}{dt}f(q,p) = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial p}(-\frac{\partial H}{\partial q}) = \{f, H\} \tag{4}$$

If f also depends on time then

$$\frac{d}{dt}f(q,p,t) = \{f, H\} + \frac{\partial f}{\partial t} \tag{5}$$

1.1 Properties of Poisson brackets

The following properties follow from the definition of Poisson brackets:

(a)

$$\{f, g\} = -\{g, f\} \tag{6}$$

In particular, this implies that

$$\{f, f\} = 0 \tag{7}$$

(b)

$$\{f, g+h\} = \{f, g\} = \{f, h\}, \quad \{f+g, h\} = \{f, h\} + \{g, h\} \tag{8}$$

(c) From the chain rule for partial derivatives we can check that

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \quad \{fg, h\} = f\{g, h\} + \{f, h\}g \tag{9}$$

(d) A very important property, which can be checked with some effort, is

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{10}$$

In quantum mechanics, we will have

$$\{f,g\} \;\to\; i[\hat{f},\hat{g}] \tag{11}$$

and we can see that the above properties become natural properties of quantum operators.

2 Canonical transformations

The dynamics of a classical system is obtained by requiring that

$$S=\int_{initial}^{final} Ldt \tag{12}$$

is extremized. But in this process of extremization, we have to keep fixed the values of the dynamical coordinates q at the endpoints. Suppose we change the Lagrangian as follows

$$L \;\to\; L+\frac{dF(q,t)}{dt} \tag{13}$$

where F is an arbitrary function of q,t. While the Lagrangian changes, the action S changes in a simple way

$$S \;\to\; S+F(q(t_f),t_f)-F(q(t_i),t_i) \tag{14}$$

But this new action will give the same dynamics as the old action, since in variations of S the extra terms would have no variation anyway. So we see that we have a freedom of adding a total time derivative to our Lagrangian.

How does this freedom show up in the Hamiltonian description? Now the dynamical variables are q,p. The Lagrangian can be written as

$$L=p\dot{q}-H \tag{15}$$

Thus the action is

$$S=\int pdq-Hdt \tag{16}$$

We can change this to

$$S=\int pdq-Hdt+dF \tag{17}$$

without changing the dynamics.

But what exactly is the meaning of the phrase: 'without changing the dynamics'? In the Lagrangian description, we had the freedom of a function $F(q,t)$. In this case we would write

$$\int pdq-Hdt \;\to\; \int pdq-Hdt+F_{,q}dq+F_{,t}dt \tag{18}$$

Comparing the left and right sides, we see that

$$q\to q, \quad p\to p+F_{,q}, \quad H\to H-F_{,t} \tag{19}$$

Thus the dynamical coordinate q does not change, and this is the only quantity which mattered in the Lagrangian description. So when we said the dynamics did not change, we meant that we have the same variable q, and it satisfies the same Euler-Lagrange equations.

But if we use Hamiltonians, then we can be more adventurous in our choice of F. We can make it depend not only on q, t but also on p. What will be the change in description upon using such an F?

Now we will write

$$\int pdq - Hdt \;\; \rightarrow \;\; \int pdq - Hdt + F_{,q}dq + F_{,p}dp + F_{,t}dt \tag{20}$$

But now we see that we cannot compare variables on the two sides: the LHS has a term in dq, while the RHS has two terms in dq and one term in dp. It would seem that we need $F_{,p} = 0$, so we are back where we started, with $F = F(q, t)$.

But we can get more general F by letting the new coordinates Q be different from the old coordinates q, and this given general canonical transformations.

3 A simple example about making a canonical transformation

Let

$$Q = q^2 \tag{21}$$

Then we see that

$$\{q^2, \frac{p}{2q}\} = 1 \tag{22}$$

so we can take

$$P = \frac{p}{2q} \tag{23}$$

Now suppose that we took

$$Q = q^2 + p \tag{24}$$

Now it is not so clear what we should take for P. Let us use a generating function $F(q, Q)$. Then we have

$$p = F_{,q} = Q - q^2 \tag{25}$$

which gives

$$F = Qq - \frac{q^3}{3} + g(Q) \tag{26}$$

Then we will have

$$P = -F_{,Q} = -q - g'(Q) = -q + h(q^2 + p) \tag{27}$$

where h is an arbitrary function of its argumnent.

4 Problem 9.8 Goldstein

Take $F(q_1, q_2, Q_1, Q_2)$. Then

$$p_1 = F_{q_1}, \quad P_1 = -F_{Q_1} \tag{28}$$

First, we try to use variables q_i, Q_i. Let us see if this is possible. It will not be, since $Q_1 = q_1$, so these variables are not independent. We cannot take

$$q_1, P_1, q_2, P_2 \tag{29}$$

either, since P_2 is given in terms of q_1, q_2. We cannot use

$$p_1, p_2, Q_1, Q_2 \tag{30}$$

since $Q_2 = p_2$. We also cannot use p_i, P_i since P_1 is given in terms of p_i.

So we have to use a mixed representation. We can take

$$q_1, P_1, q_2, Q_2 \tag{31}$$

Then we will have

$$p_1 = F_{,q_1}, \quad Q_1 = F_{,P_1}, \quad p_2 = F_{,q_2}, \quad P_2 = -F_{,Q_2} \tag{32}$$

Let us write all LHS variables in terms of the needed ones. We have

$$p_1 = P_1 + 2Q_2, \quad Q_1 = q_1, \quad p_2 = Q_2, \quad P_2 = -2q_1 - q_2 \tag{33}$$

Then our equations are

$$P_1 + 2Q_2 = F_{,q_1} \to F = q_1(P_1 + 2Q_2) + f(P_1, q_2, Q_2) \tag{34}$$

$$q_1 = F_{,P_1} = q_1 + f_{,P_1} \to f = f(q_2, Q_2) \tag{35}$$

$$Q_2 = F_{,q_2} = f_{,q_2}, \quad f = q_2 Q_2 + g(Q_2) \tag{36}$$

$$-2q_1 - q_2 = -F_{,Q_2} = -2q_1 - q_2 + g'(Q_2) \tag{37}$$

This gives $g = 0$, and we have

$$F = q_1(P_1 + 2Q_2) + q_2 Q_2 \tag{38}$$