1 Quantizing the open string

1.1 The action

The Polyakov action is

$$S = -T \int d^2\xi \sqrt{-g} \frac{1}{2} \partial_a X^\mu \partial_b X_\mu g^{ab} \tag{1}$$

In the gauge

$$g_{ab} = \eta_{ab} \tag{2}$$

we get

$$S = T \int d\tau d\sigma \frac{1}{2} [\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu] \tag{3}$$

We will let the range of σ be

$$0 \le \sigma < \pi \tag{4}$$

1.2 Different boundary conditions

Neumann boundary conditions (N) correspond to X^μ having a vanishing derivative

$$\frac{\partial X^\mu}{\partial \sigma} = 0 \tag{5}$$

Dirichlet boundary conditions (D) correspond to X^μ being fixed

$$X^\mu = x_0^\mu \tag{6}$$

Let us consider some simple cases:

(a) NN boundary conditions:

We can expand the coordinates as

$$X^\mu = \sum_{n=0}^{\infty} f_n(\tau) \cos n\sigma \tag{7}$$

We see that

$$\sigma = 0 : \quad \frac{d}{d\sigma} \cos n\sigma = -n \sin n\sigma = 0 \tag{8}$$

$$\sigma = \pi : \quad \frac{d}{d\sigma} \cos n\sigma = -n \sin n\sigma = 0 \tag{9}$$

(b) DD boundary conditions:

Suppose that both endpoints of the string are at $X^\mu = 0$. We can expand the coordinates as

$$X^\mu = \sum_{n=0}^{\infty} f_n(\tau) \sin n\sigma \tag{10}$$

We see that

$$\sigma = 0: \quad X^\mu \sim \sin n\sigma = 0 \tag{11}$$

$$\sigma = \pi: \quad X^\mu \sim \sin n\sigma = 0 \tag{12}$$

If X^μ is not zero at both ends, then we write

$$X^\mu = a^\mu + b^\mu \sigma + \sum_{n=0}^{\infty} f_n(\tau) \sin n\sigma \tag{13}$$

where a^μ, b^μ are not variables but constants which are determined to give the two endpoints of the open string.

(c) ND boundary conditions:

We can expand the coordinates as

$$X^\mu = \sum_{n=0}^{\infty} f_n(\tau) \cos(n + \frac{1}{2})\sigma \tag{14}$$

We see that

$$\sigma = 0: \quad \frac{d}{d\sigma} \cos(n + \frac{1}{2})\sigma = -(n + \frac{1}{2}) \sin(n + \frac{1}{2})\sigma = 0 \tag{15}$$

$$\sigma = \pi: \quad X^\mu \sim \cos(n + \frac{1}{2})\sigma = 0 \tag{16}$$

Here we assumed that $X^\mu = 0$ at the endpoint with Dirichlet boundary condition. If this is not the case we write

$$X^\mu = a^\mu + \sum_{n=0}^{\infty} f_n(\tau) \cos(n + \frac{1}{2})\sigma \tag{17}$$

where a^μ is a constant vector and not a variable.

(d) DN boundary conditions:

We can expand the coordinates as

$$X^\mu = \sum_{n=0}^{\infty} f_n(\tau) \sin(n + \frac{1}{2})\sigma \tag{18}$$

We see that

$$\sigma = 0: \quad X^\mu \sim \cos(n + \frac{1}{2})\sigma = 0 \tag{19}$$

$$\sigma = \pi: \quad \frac{d}{d\sigma} \cos(n + \frac{1}{2})\sigma = -(n + \frac{1}{2}) \sin(n + \frac{1}{2})\sigma = 0 \tag{20}$$

Here we assumed that $X^\mu = 0$ at the endpoint with Dirichlet boundary condition. If this is not the case we write

$$X^\mu = a^\mu + \sum_{n=0}^{\infty} f_n(\tau) \sin(n + \frac{1}{2})\sigma \tag{21}$$

where a^μ is a constant vector and not a variable.

2 Expanding in oscillators

Let us take the case of NN boundary conditions. Let us make normalized modes out of the spatial functions. We have

$$\int_0^\pi d\sigma = \pi \tag{22}$$

$$\int_0^\pi d\sigma \cos^2 n\sigma = \frac{\pi}{2} \quad (n > 0) \tag{23}$$

Thus the normalized spatial modes are

$$n = 0: \quad \frac{1}{\sqrt{\pi}} \tag{24}$$

$$n > 0: \quad \sqrt{\frac{2}{\pi}} \cos n\sigma \tag{25}$$

Thus expand the coordinates as

$$X^\mu(\tau, \sigma) = f_0(\tau) \frac{1}{\sqrt{\pi}} + \sum_{n>0} f_n(\tau) \sqrt{\frac{2}{\pi}} \cos n\sigma \tag{26}$$

Put this in the action. Then we get

$$S = T[\frac{1}{2}(\dot{f}_0)^2 + \sum_{n>0} \frac{1}{2}(\dot{f}_n^2 - n^2 f_n^2)] \tag{27}$$

Thus the variable

$$q^\mu \equiv \sqrt{T} f_0 \tag{28}$$

behaves as a free particle variable with Lagrangian

$$\frac{1}{2}\dot{q}^2 \tag{29}$$

while the variables

$$q_n^\mu = \sqrt{T} f_n, \quad n > 0 \tag{30}$$

behave as harmonic oscillators with Lagrangian

$$\frac{1}{2}[(\dot{q}_n^\mu)^2 - (q_n^\mu)^2] \tag{31}$$

The oscillators have angular frequency

$$\omega = n \tag{32}$$

Thus let us recall how a harmonic oscillator is quantized. We have

$$a = \frac{1}{\sqrt{2}}(\sqrt{\omega}x + i\frac{p}{\sqrt{\omega}}), \quad a^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}x - i\frac{p}{\sqrt{\omega}}) \tag{33}$$

Then using $[x, p] = i$, we get

$$[a, a^\dagger] = 1 \tag{34}$$

Note that

$$x = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \tag{35}$$

In the Heisenberg picture of evolution we write

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(\hat{a}e^{-i\omega\tau} + \hat{a}^\dagger e^{i\omega\tau}) \tag{36}$$

Returning to our system, we see that we should write for $n > 0$

$$q_n^\mu = \frac{1}{\sqrt{2n}}(\hat{a}^\mu e^{-in\tau} + \hat{a}^{\mu\dagger} e^{in\tau}) \tag{37}$$

In the expansion of the x^μ these nonzero modes will therefore give

$$X^\mu \Rightarrow \sum_{n>0} \sqrt{\frac{2}{\pi}} \cos n\sigma \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2n}}(a_n^\mu e^{-in\tau} + a_n^{\mu\dagger} e^{in\tau}) \tag{38}$$

This gives

$$X^\mu \Rightarrow \sum_{n>0} \sqrt{2\alpha'} \frac{1}{\sqrt{n}}(a_n^\mu e^{-in\tau} + a_n^{\mu\dagger} e^{in\tau}) \tag{39}$$

Now define

$$\alpha_n^\mu = -i\sqrt{n}a_n^\mu, \quad n > 0 \tag{40}$$

$$\alpha_{-n}^\mu = i\sqrt{n}a_n^{\mu\dagger}, \quad n > 0 \tag{41}$$

Then we will get

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0} \tag{42}$$

and the expansion becomes

$$X^\mu \Rightarrow i \sum_{n\neq 0} \sqrt{2\alpha'} \cos n\sigma \frac{\alpha_n^\mu}{n} e^{-in\tau} \tag{43}$$

We still have to tackle the zero mode. We write it in a way that will allow us to add α_0^μ to the other α_n^μ

$$X^\mu = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau \tag{44}$$

Recall that $X^\mu = f_0/\sqrt{\pi}$, and $q^\mu = \sqrt{T} f_0$. Thus

$$X^\mu \Rightarrow \frac{1}{\sqrt{T}\sqrt{\pi}}(q^\mu) \tag{45}$$

Let us expand the zero mode q^μ

$$X^\mu \Rightarrow \frac{1}{\sqrt{T}\sqrt{\pi}}(Q^\mu + \Pi_q^\mu \tau) \tag{46}$$

We will write

$$x_0^\mu = \frac{q_0^\mu}{\sqrt{T}\sqrt{\pi}} = \sqrt{2\alpha'}q_0^\mu \tag{47}$$

Since $[Q^\mu, \Pi^\nu] = i\eta^{\mu\nu}$, and we want to get

$$[x_0^\mu, p^\nu] = \eta^{\mu\nu} \tag{48}$$

we see that we must take

$$p^\mu = \frac{\Pi^\mu}{2\alpha'} = \frac{\alpha_0^\mu}{\sqrt{2\alpha'}} \tag{49}$$

The full mode expansion for the open string therefore becomes

$$X^\mu = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \sum_{n\neq 0}\sqrt{2\alpha'}\cos n\sigma\frac{\alpha_n^\mu}{n}e^{-in\tau} \tag{50}$$

where

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu \tag{51}$$

3 The Virasoro constraints

The conditions are

$$\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu = 0 \tag{52}$$

This gives

$$0 = \sum_{m,n}\alpha_m^\mu\alpha_{\mu,n}(\sin n\sigma \sin m\sigma - \cos n\sigma\cos m\sigma)e^{i(n-m)\tau} = \sum_{m,n}\alpha_m^\mu\alpha_{\mu,n}\cos(n-m)\sigma e^{i(n-m)\tau} \tag{53}$$

Thus for all fourier modes of the constraint to vanish we need

$$L_n = \frac{1}{2}\alpha_m^\mu\alpha_{\mu,n-m} = 0 \tag{54}$$

The L_0 constraint again will be

$$(L_0 - 1)|\psi\rangle = 0 \tag{55}$$

We have

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0}\alpha_{-n}\alpha_n = \alpha' p^2 + \sum_{n>0}\alpha_{-n}\alpha_n \tag{56}$$

Thus we find that the mass of a state is given by

$$m^2 = -p^2 = \frac{1}{\alpha'}(N-1) \tag{57}$$

where N is the level of the state.

The lowest mode is therefore a tachyon with

$$N = 0, \quad m^2 = -\frac{1}{\alpha'} \tag{58}$$

The next set of excitations are massless

$$N = 1, \quad m^2 = 0 \tag{59}$$

and give the massless photon. The next level gives

$$N = 2, \quad m^2 = \frac{1}{\alpha'} \tag{60}$$

We can gauge away the C_{0i} which kills the D_i, as we show below. D_0 is determined by the $C_{\mu\nu}$. The $C_{\mu\nu}$ are symmetric with one condition on them, so we have

$$\frac{(D-1)D}{2} - 1 = \frac{(D-2)(D-1)}{2} \tag{61}$$

states.

4 Gauging away states

Let us now check that we can indeed gauge away the states claimed above. We had derived the physical state constraints at level 2 in the closed string case, so let us work with the closed string for the moment; the analysis is essentially the same for open strings.

To show that C_{0i} can be gauged away, consider the states

$$L_{-1}|\lambda\rangle \tag{62}$$

where $|\lambda\rangle$ is a state at level one

$$|\lambda\rangle = F_\mu \alpha^\mu_{-1}|0\rangle \tag{63}$$

The relevant part of L_{-1} is

$$L_{-1} = \alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + \dots \tag{64}$$

We get

$$L_{-1}|\lambda\rangle = (\alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1)F_\mu\alpha^\mu_{-1}|0\rangle = \frac{p_\nu}{\sqrt{2}}F_\mu\alpha^\nu_{-1}\alpha^\mu_{-1}|0\rangle + F_\mu\alpha^\mu_{-2}|0\rangle \tag{65}$$

In the frame where we have $p^\mu = (2, 0, \dots 0)$ we see that we can get any value for $C_{0\mu}$ by choosing F^μ appropriately. We automatically get a corresponding D_μ, with

$$D_\mu = \frac{1}{\sqrt{2}}(-2)C_{0\mu} = -\sqrt{2}C_{0\mu} \tag{66}$$

Note that this is exactly one of the conditions satisfied by $C_{\mu\nu}$ and D_μ for physical states.

But while we can make states like (63) which are orthogonal to all physical states, we are looking at present for such states which are also physical states; only then will they be pure gauge modes. Thus we should check if (63)satisfies the physical state conditions. First we should check if

$$0 = L_1 L_{-1}|\lambda\rangle = 2L_0|\lambda\rangle + L_{-1}L_1|\lambda\rangle \tag{67}$$

Note that

$$L_0|\lambda\rangle = 0 \tag{68}$$

since $L_0 = 1$ on the final state that we construct. We need to ensure that

$$0 = L_1|\lambda\rangle = \alpha_0\alpha_1 F_\mu \alpha^\mu_{-1}|0\rangle = \frac{p^\mu}{\sqrt{2}}F_\mu \tag{69}$$

Thus in our rest frame we need that

$$F^0 = 0 \tag{70}$$

We must also check that

$$0 = L_2L_{-1}|\lambda\rangle = 3L_1|\lambda\rangle = 3\alpha_0\alpha_1 F^\mu\alpha_{-1,\mu}|0\rangle = 3\frac{p^\nu}{\sqrt{2}}F_\nu|0\rangle \tag{71}$$

In the rest frame we again see that we need $F^0 = 0$, and we thus get a physical state with this condition.

Thus we have been able to make as pure gauge states the states which have arbitrary F^i, which means that we can get arbitrary $C_{0i} \neq 0$. We can add these pure gauge states to any physical state to get a physical state which has

$$C_{0i} = 0 \tag{72}$$

So we can indeed make physical states where C_{0i} are gauged away.

Before proceeding, recall that the pure gauge state that we have made is of the kind that we discussed in the last set of notes:

$$|\psi\rangle = L_{-1}|\lambda\rangle, \qquad L_n|\lambda\rangle = 0 \;\; for \; n > 0 \tag{73}$$

Let us now see if we can get another pure gauge state in the same manner that we obtained there are level 2. Thus we try

$$(L_{-2} + \gamma L^2_{-1})|0\rangle \tag{74}$$

Note that

$$L_n|0\rangle = 0, \qquad n > 0 \tag{75}$$

So we know from our general analysis that we must have $\gamma = \frac{3}{2}$ and $D = 26$. Thus the state will have the form

$$(L_{-2} + \frac{3}{2}L^2_{-1})|0\rangle \tag{76}$$

Let us compute this explicitly. The relevant part of L_{-2} is

$$L_{-2} = \frac{1}{2}\alpha_{-1}\alpha_{-1} + \alpha_{-2}\alpha_0 + \ldots \tag{77}$$

Thus creates

$$L_{-2}|0\rangle = \frac{p_\mu}{\sqrt{2}}\alpha^\mu_{-2}|0\rangle + \frac{1}{2}\alpha^\mu_{-1}\alpha_{\mu,-1}|0\rangle \tag{78}$$

The first L_{-1} generates

$$L-1|0\rangle = \alpha^\mu_{-1}\alpha_{\mu,0}|0\rangle = \frac{p_\mu}{\sqrt{2}}\alpha^\mu_{-1}|0\rangle \tag{79}$$

For the next application of L_{-1} we need the following terms in L_{-1}

$$L_{-1} = \alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + \dots \tag{80}$$

We find

$$L_{-1}^2|0\rangle = \frac{p_\mu p_\nu}{2}\alpha^\mu_{-1}\alpha^\nu_{-1} + \frac{p_\mu}{\sqrt{2}}\alpha^\mu_{-2}|0\rangle \tag{81}$$

Overall we get

$$(L_{-2} + \frac{3}{2}L_{-1}^2)|0\rangle = \frac{5}{2}\frac{p_\mu}{\sqrt{2}}\alpha^\mu_{-2}|0\rangle + \frac{1}{2}\alpha^\mu_{-1}\alpha_{\mu,-1} + \frac{3}{4}p_\mu p_\nu \alpha^\mu_{-1}\alpha^\nu_{-1} \tag{82}$$

In our general form of the state

$$C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle + D_\mu \alpha^\mu_{-2}|0\rangle \tag{83}$$

we find that our pure gauge state will be of the form

$$C_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu} + \frac{3}{4}p_\mu p_\nu \tag{84}$$

$$D_\mu = \frac{5}{2}\frac{p_\mu}{\sqrt{2}} \tag{85}$$

Recall that the constrains from $L_1|\psi\rangle = 0, L_2|\psi\rangle = 0$ were respectively

$$\frac{1}{\sqrt{2}}C_{\mu\nu}p^\nu + D_\mu = 0 \tag{86}$$

$$C^\mu{}_\mu + \sqrt{2}p^\mu D_\mu = 0 \tag{87}$$

Note that $p^2 = -4$, and in the rest frame, $p = (2, 0, 0, \dots 0)$. We can then check that both these conditions are satisfied. Consider the first one. We get

$$\frac{1}{\sqrt{2}}[\frac{p_\mu}{2} + \frac{3}{4}(-4)p_\mu] + \frac{5}{2}\frac{p_\mu}{\sqrt{2}} = 0 \tag{88}$$

The second condition gives

$$(\frac{D}{2} + \frac{3}{4}(-4) + \sqrt{2}(-4)\frac{1}{\sqrt{2}}\frac{5}{2}) = \frac{D}{2} - 3 - 10 = 0 \tag{89}$$

where we have used that $D = 26$.

So we indeed have another physical state which is pure gauge from here. Let us now get the true degrees of freedom in a convenient gauge. We have already seen that C_{0i} can be set to zero, and this makes $D_i = 0$. We are working in the rest frame. Using the pure gauge state (78) we see that we can set $D_0 = 0$. From the first physical condition (86) we then get

$$C_{00} = 0 \tag{90}$$

Thus we are left only with the components C_{ij}. The second physical condition (87) then gives

$$C_{ii} = 0 \tag{91}$$

So our degrees of freedom correspond to a traceless symmetric tensor whose components are orthogonal to the momentum of the particle.