

1 Modular invariance

In string theory we have a 2-d world sheet. The path integral over this world sheet gives a quantity Z . But we can compute Z in a Hamiltonian formulation as well, which needs that we choose one direction as space and another as time. Of course both choices should give the same answer, but this is not manifestly obvious from the computation. Checking this equality needs interesting mathematical identities. The physics of modular invariance provides deep properties of string theory: consistency conditions and dualities.

2 The 1-loop path integral

The Hamiltonian is given by the energy, which is given by

$$E = \int d\Sigma^\mu T_{\mu\nu} \xi^\nu \quad (1)$$

where $d\Sigma^\mu$ is the volume element of a surface and ξ^ν is the timelike killing vector of the system. The metric is

$$ds^2 = -d\tau^2 + d\sigma^2 \quad (2)$$

We have

$$d\Sigma^\tau = d\sigma \quad (3)$$

$$\xi^\tau = 1, \quad \xi^\sigma = 0 \quad (4)$$

Thus we have

$$E = \int_0^{2\pi} d\sigma T_{\tau\tau} \quad (5)$$

But

$$T_{\tau\tau} = T_{zz} \frac{\partial z}{\partial \tau} \frac{\partial z}{\partial \tau} + T_{\bar{z}\bar{z}} \frac{\partial \bar{z}}{\partial \tau} \frac{\partial \bar{z}}{\partial \tau} = T_{zz} + T_{\bar{z}\bar{z}} \quad (6)$$

where we have used that

$$z = t_E + i\sigma, \quad \bar{z} = t_E - i\sigma \quad (7)$$

where T_E is Euclidean time. On the plane we had defined

$$L_n = \int_C dz T_{zz} z^{n+1} \quad (8)$$

We go to the cylinder via the map

$$z = e^w, \quad \frac{\partial z}{\partial w} = e^w \quad (9)$$

Thus

$$L_n = \int_C \frac{\partial z}{\partial w} dw T_{ww} \left(\frac{\partial w}{\partial z}\right)^2 z^{n+1} = \int_C dw T_{ww} e^{nw} \quad (10)$$

where for the moment we have ignored the anomaly in the transformation. In particular,

$$L_0 = \int_0^{2\pi} d\sigma T_{ww} \quad (11)$$

Thus

$$E = L_0 + \bar{L}_0 \quad (12)$$

The partition function will be

$$Z = \sum_n e^{-E_n T_E} \quad (13)$$

where our torus extends from $t_E = 0$ to T_E . We will assume for now that the space and time directions of the torus are at right angles to each other. The shape of the torus is specified by writing

$$\tau = i \frac{T_E}{2\pi} \quad (14)$$

where T_E is the length in the time direction and 2π is the length of the space direction.

Now let us return to the anomaly. On the plane, we had

$$L_0|0\rangle = 0 \quad (15)$$

On the cylinder we should get the same result by definition, but we find that

$$\langle T_{ww} \rangle = -\frac{c}{24} \quad (16)$$

Thus we should write on the cylinder

$$L_0 = \int dw (T_{ww} + \frac{c}{24}) \quad (17)$$

and then we will get L_0 to annihilate the vacuum.

But now let us ask what it is that we wanted to compute on the cylinder. We wanted to weight each state with its energy on the cylinder, and the vacuum has a negative Casimir energy contribution $-\frac{c}{24}$. Thus we should not assign $E = 0$ to the vacuum, rather we should assign $-\frac{c}{24}$. Thus we have to compute

$$Z = \sum_n e^{-2\pi\tau(-i)[(L_0 - \frac{c}{24}) + (\bar{L}_0 - \frac{c}{24})]} = \sum_n e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{2\pi i\tau(\bar{L}_0 - \frac{c}{24})} \quad (18)$$

Let us define

$$q = e^{2\pi i\tau} \quad (19)$$

Thus we have to compute

$$\sum_n q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \quad (20)$$

where these L_0, \bar{L}_0 are defined so that they annihilate the vacuum.

3 Computing L_0

We have seen that

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{n>0} \alpha_{-n} \alpha_n \quad (21)$$

Consider any one oscillator mode α_n, α_{-n} . The occupation number in this mode can be $0, 1, 2, \dots$. The state with occupation number m will contribute to L_0

$$\alpha_{-n}\alpha_n|m\rangle = na_n^\dagger a_n|m\rangle = nm \quad (22)$$

The contribution to the path integral from these occupation numbers will be

$$\sum_{m=0}^{\infty} q^{nm} = \frac{1}{1-q^n} \quad (23)$$

There are D such oscillators. Thus from all these occupation modes we will get the contribution

$$\left[\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right]^D \quad (24)$$

From the right moving oscillators we will get

$$\left[\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n} \right]^D \quad (25)$$

The zero modes will give

$$\int d^D p q^{\frac{\alpha'}{2} p^2} = \int d^D p e^{2\pi i \tau \frac{\alpha'}{2} p^2} = \left[\sqrt{\frac{\pi}{-2\pi i \frac{\alpha'}{2} \tau}} \right]^D = \left[\frac{i}{\alpha' \tau} \right]^{\frac{D}{2}} \quad (26)$$

We must now include the factor $q^{-\frac{c}{24}} \bar{q}^{-\frac{c}{24}}$, noting that each boson has $c = 1$. The overall partition function will be

$$Z = \left[\left(\frac{i}{\alpha' \tau} \right)^{\frac{1}{2}} q^{-\frac{1}{24}} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right) \bar{q}^{-\frac{1}{24}} \left(\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n} \right) \right]^D \equiv Z_1^D \quad (27)$$

where

$$Z_1 = \left(\frac{i}{\alpha' \tau} \right)^{\frac{1}{2}} q^{-\frac{1}{24}} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right) \bar{q}^{-\frac{1}{24}} \left(\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n} \right) \quad (28)$$

is the partition function for one boson.

4 Theta functions

The theta functions have the basic form

$$\sum_{n=-\infty}^{\infty} e^{-\mu n^2} \quad (29)$$

We will write

$$\Theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau n^2 + 2\pi i n z} \quad (30)$$

These functions have two main properties. One is that they have product representations. The other is that they satisfy modular transformation properties. Let us look at the second property first. This arises from the Poisson resummation formula, which we now describe.

4.1 The Poisson resummation formula

Suppose we have the sum

$$Z(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} \quad (31)$$

If we see n^2 in the exponent, we can imagine that it got there by a Gaussin integral

$$\int_{x=-\infty}^{\infty} dx e^{-\alpha x^2 + i\beta x} = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\beta^2}{4\alpha}} \quad (32)$$

Thus we set

$$4\alpha = -\frac{1}{i\pi\tau}, \quad \alpha = \frac{i}{4\pi\tau} \quad (33)$$

and then use

$$\int_{x=-\infty}^{\infty} dx e^{-\frac{i}{4\pi\tau}x^2 + inx} = \sqrt{-4i\pi^2\tau} e^{i\pi\tau n^2} \quad (34)$$

This gives

$$Z(\tau) = \sum_{n=-\infty}^{\infty} \int_{x=-\infty}^{\infty} dx \frac{1}{\sqrt{-4i\pi^2\tau}} e^{-\frac{i}{4\pi\tau}x^2 + inx} \quad (35)$$

Let us do the sum first. We have

$$\sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} 2\pi\delta(x - 2\pi m) \quad (36)$$

The periodic nature of the LHS is evident, so we get a sum of delta functions. In any interval like $0 \leq x < 2\pi$, we have

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi\delta(x) \quad (37)$$

The normalization follows upon integrating both sides in x . The LHS gives 2π from the term $n = 0$ and no contribution from the other terms.

Thus we have

$$Z(\tau) = \int_{x=-\infty}^{\infty} dx \sum_{m=-\infty}^{\infty} 2\pi\delta(x - 2\pi m) e^{-\frac{i}{4\pi\tau}x^2} \quad (38)$$

Doing the x integral sets $x = 2\pi m$. So have

$$Z(\tau) = \sum_{m=-\infty}^{\infty} \frac{2\pi}{\sqrt{-4i\pi^2\tau}} e^{-\frac{i}{2\pi\tau}4\pi^2 m^2} = \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{-i\tau}} e^{-\frac{i}{\tau}\pi m^2} \quad (39)$$

Thus we have found that

$$Z(\tau) = \frac{1}{\sqrt{-i\tau}} Z\left(-\frac{1}{\tau}\right) \quad (40)$$

5 Sum and product representations

We had defined

$$\Theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2 + 2\pi i n z} \quad (41)$$

We write

$$q = e^{2\pi i \tau} \quad (42)$$

The theta function has a product representation

$$\Theta(z, \tau) = \prod_{m=1}^{\infty} (1 - q^m)(1 + e^{2\pi i z} q^{m-\frac{1}{2}})(1 + e^{-2\pi i z} q^{m-\frac{1}{2}}) \quad (43)$$

We can also define ‘theta functions with characteristics’

$$\Theta \begin{pmatrix} a \\ b \end{pmatrix} (z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi(n+a)^2\tau + 2\pi i(n+a)(z+b)} \quad (44)$$

We can look at some special cases:

$$\Theta \begin{pmatrix} 0 \\ 0 \end{pmatrix} \equiv \Theta_{00} \equiv \Theta_3 \quad (45)$$

$$\Theta \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \equiv \Theta_{01} \equiv \Theta_4 \quad (46)$$

$$\Theta \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \equiv \Theta_{10} \equiv \Theta_2 \quad (47)$$

$$\Theta \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \equiv \Theta_{11} \equiv -\Theta_1 \quad (48)$$

The function we encounter most commonly will be

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \left[\frac{\partial_z \Theta_{11}(0, \tau)}{-2\pi} \right]^{\frac{1}{3}} \quad (49)$$

The transformation property is

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau) \quad (50)$$

We can now check that the partition function of the boson is modular invariant. We have

$$Z_1(\tau) = \left(\frac{i}{\alpha'\tau}\right)^{\frac{1}{2}} |\eta(\tau)|^{-2} \quad (51)$$

Thus

$$Z_1\left(-\frac{1}{\tau}\right) = \left(-\frac{i\tau}{\alpha'}\right)^{\frac{1}{2}} \left|\eta\left(-\frac{1}{\tau}\right)\right|^{-2} \quad (52)$$

Using the transformation property of η , we have

$$Z_1(-\frac{1}{\tau}) = (-\frac{i\tau}{\alpha'})^{\frac{1}{2}} |(-i\tau)|^{-1} |\eta(\tau)|^{-2} \quad (53)$$

Note that we have taken $\tau = i\tau_2$ to be pure imaginary in the above discussion. Thus we have

$$Z_1(-\frac{1}{\tau}) = (\frac{\tau_2}{\alpha'})^{\frac{1}{2}} \tau_2^{-1} |\eta(\tau)|^2 = (\frac{1}{\alpha'\tau_2})^{\frac{1}{2}} |\eta(\tau)|^2 = (\frac{i}{\alpha'\tau})^{\frac{1}{2}} |\eta(\tau)|^2 \quad (54)$$

Thus we see that

$$Z(-\frac{1}{\tau}) = Z(\tau) \quad (55)$$

Note that if we dropped any mode from the boson, we would not get the modular invariance.

6 Proving the relation between sum and product representations

Let us consider a fermion field in 1+1 dimensions, in a finite box. Then we have a one parameter set of discrete energy levels. The negative energy levels are filled to make the fermi sea, and the positive energy levels are empty in this vacuum configuration. One may wonder whether to fill the zero energy level, but we will not have such a level: we will let the levels be

$$\dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots \quad (56)$$

If we create a fermion in the level $\frac{1}{2}$ then it will have energy

$$E = \frac{1}{2} \quad (57)$$

and charge

$$L = 1 \quad (58)$$

since we have added one fermion to the system. If we create a hole in the level $-\frac{1}{2}$ then we will again have energy

$$E = \frac{1}{2} \quad (59)$$

but a charge

$$L = -1 \quad (60)$$

since we have deleted a fermion from the system. We can make states with any net energy and charge. We will compute the partition function by writing

$$Z = \sum_n q^{E_n} \mu^{L_n} \quad (61)$$

where the sum runs over all states. We will compute this sum in two ways, and equate the results to get the desired identity.

6.1 First method

Consider a positive energy level with $E = n - \frac{1}{2}$. This level can be unoccupied or occupied. The first gives a contribution 1, while the second gives $q^{n-\frac{1}{2}}\mu$. Thus overall we get

$$(1 + q^{n-\frac{1}{2}}\mu) \quad (62)$$

From all the positive energy levels we thus get

$$\prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}\mu) \quad (63)$$

Now consider a negative energy level $E = -(n - \frac{1}{2})$. If it is occupied, we get 1. If it is unoccupied, we get $q^{n-\frac{1}{2}}\mu^{-1}$. From all these levels we will get

$$\prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}\mu^{-1}) \quad (64)$$

Thus the total partition function is

$$Z = \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}\mu)(1 + q^{n-\frac{1}{2}}\mu^{-1}) \quad (65)$$

6.2 Second method

We start with the vacuum, and first make the lowest energy state that has charge L . Suppose L is positive. Then we just add L fermions in the levels $\frac{1}{2}, \frac{3}{2}, \dots, L - \frac{1}{2}$. If L is negative, we remove fermions from the highest L levels. We call the resulting state the vacuum in the charge sector L .

Now focus on one such charge sector. We want to see all states in this sector and record their energies. The vacuum itself has an energy

$$E = \frac{1}{2} + \frac{3}{2} + \dots + L - \frac{1}{2} = \frac{L(L+1)}{2} - \frac{L}{2} = \frac{L^2}{2} \quad (66)$$

Now let us see how other states with higher energy can be constructed. Suppose we want to raise the energy further by E_1 . Then this energy can go towards raising some set of fermions to higher energy levels. Suppose we take the highest fermion and move it up by λ_1 units. We move the next fermion up by λ_2 units. We cannot take $\lambda_2 > \lambda_1$, because if we did that then the fermions would either land up in the same level (for $\lambda_2 = \lambda_1 + 1$), which is not allowed by the Pauli principle, or the second fermion would end up higher than the first, and give us a double counting of states (the fermions are indistinguishable, so we do not want to count as distinct the possibilities where one or other fermion was the highest one). Thus we need

$$\lambda_1 \geq \lambda_2 \geq \dots \lambda_k \quad (67)$$

where we moved k fermions in all. Further, we must have

$$\sum_k \lambda_k = E_1 \quad (68)$$

Thus for each partition of E_1 we get a new state. Thus the number of states is

$$p(E_1) \tag{69}$$

where $p(n)$ is the number of partitions of n . The contribution to the partition function from this sector will be

$$Z \rightarrow q^{\frac{L^2}{2}} p(E_1) q^{E_1} \mu^L \tag{70}$$

But the count of states with extra energy E_1 is the same for all charge sectors L . Thus we will have

$$Z = \left(\sum_{E_1=0}^{\infty} p(E_1) q^{E_1} \right) \left(\sum_{L=-\infty}^{\infty} \mu^L q^{\frac{L^2}{2}} \right) \tag{71}$$

Now we will check that

$$\sum_{E_1=0}^{\infty} p(E_1) q^{E_1} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{72}$$

To prove this, note that

$$\frac{1}{1-q^n} = 1 + q^n + q^{2n} + q^{3n} + \dots \tag{73}$$

Look for a power of q^{E_1} in $\prod_{n=1}^{\infty} \frac{1}{1-q^n}$. Suppose that from the expansion (73) we pick the term s_n . Then we will say that the term n appears s times in the partition of E_1 . Overall if we have $s_i n_i$ from several different factors $\frac{1}{1-q^{n_i}}$, then we will have

$$\sum_i s_i n_i = E_1 \tag{74}$$

and the count of these possibilities gives $p(E_1)$. This proves (72).

Thus we have found

$$Z = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \left(\sum_{L=-\infty}^{\infty} \mu^L q^{\frac{L^2}{2}} \right) \tag{75}$$

6.3 Equating the two methods

Equating the two results for Z we get

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} \left(\sum_{L=-\infty}^{\infty} \mu^L q^{\frac{L^2}{2}} \right) = \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}} \mu) (1 + q^{n-\frac{1}{2}} \mu^{-1}) \tag{76}$$

or

$$\sum_{L=-\infty}^{\infty} \mu^L q^{\frac{L^2}{2}} = \prod_{n=1}^{\infty} (1 - q^n) (1 + q^{n-\frac{1}{2}} \mu) (1 + q^{n-\frac{1}{2}} \mu^{-1}) \tag{77}$$

Now setting

$$q = e^{2\pi i \tau}, \quad \mu = e^{2\pi i z} \tag{78}$$

we get

$$\Theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau n^2 + 2\pi i n z} = \prod_{m=1}^{\infty} (1 - q^m) (1 + e^{2\pi i z} q^{m-\frac{1}{2}}) (1 + e^{-2\pi i z} q^{m-\frac{1}{2}}) \tag{79}$$

as desired.

7 Exercises

(A) Prove that

$$\theta_{00}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi iz^2}{\tau}} \theta_{00}(z, \tau) \quad (80)$$

(B) Prove that

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \left[\frac{\partial_z \Theta_{11}(0, \tau)}{-2\pi} \right]^{\frac{1}{3}} \quad (81)$$