TOPIC IV

STRING THEORY

General relativity is a beautiful, complete theory of gravity. It agrees well with observations. For example it predicts the correct precession of the orbit of Mercury, and its prediction of a 'big bang' is verified by the observation of the cosmic microwave background. We will now seek to extend this theory, adding new features to reach *string theory.* But these additional features have not been observed as yet; in fact they may not be observable for several decades to come. So a natural question to ask is: why should we consider string theory?

General relativity as formulated by Einstein is a *classical* theory; it does not incorporate the ideas of quantum mechanics. Electrodynamics as formulated by Maxwell was also a classical theory, but some effort yielded a natural quantum extension, called quantum electrodynamics. Other fundamental interactions – the weak and the strong force – were also naturally obtained as fully quantum theories, with a structure entirely analogous to quantum electrodynamics. But with gravity the situation turned out to be very different, and attempts to extend gravity to a quantum theory met with repeated failure.

We will see below that this failure is due to the fact that gravity is essentially different in nature from other interactions; it is mediated by a particle – the graviton – which has spin 2 rather than spin 1. Particles of spin 2 lead to a strong divergence in quantum 'loop diagrams'; a divergence that cannot be renormalized away as is done in theories where the interaction is mediated by spin 1 particles. After some years of effort is was realized that this divergence is a robust feature of the spin 2 interaction, and fundamentally new ideas would be needed to eliminate it and get a sensible theory.

String theory is a natural and beautiful extension of general relativity, where the troublesome divergence gets automatically removed. We will see that the classical gravity theory exhibits certain discrete symmetries, called T and S duality. If we require that these symmetries persist in the full quantum theory, then we are led to a *unique* theory – string theory. The string of string theory has a natural symmetry of its own, called modular invariance, and it is this symmetry that leads to a removal of the pesky divergence.

At first it seems we have paid a heavy price for quantizing the theory in this way. We see 3+1 spacetime dimensions around us, but string theory is consistent only in 9+1 spacetime dimensions. Thus we have to imagine that 6 extra directions are curled up into circles that are too small to resolve with current experiments. Further, we have traditionally thought of elementary particles as being pointlike. In string theory we do have pointlike particles, but we also have extended objects, like strings and branes.

But remarkably, these curious features of the theory are exactly the ones that will lead to a resolution of all our puzzles with black holes. Black holes have an entropy that is much larger than the entropy of normal matter. But we will see that this entropy is exactly reproduced when the energy of the black hole is put into extended objects instead of point particles. Further, we will see that these extended objects stretch and grow in size as their energy content is increased; this makes the black hole swell into a *fuzzball*, a structure with no horizon. The fuzzball radiates energy from its surface like any other normal body, and we avoid the information paradox.

These successes with black holes are closely tied to the remarkable uniqueness of the theory. Usual theories of particle physics allow a fair bit of latitude in our choice of particle content, particle masses and interaction strengths. But in string theory properties of the elementary objects and their interactions are completely fixed; any alteration leads to a breakdown of consistency of the quantum theory. Because of this uniqueness, one gets a definite prediction for the microscopic entropy of black hole; this is now just the logarithm of the number of states of strings and branes that can be made with a given energy. This microscopic entropy is found to agree perfectly, including its overall numerical coefficient, with the Bekenstein entropy of the hole predicted by thermodynamics. A similar situation holds for the fuzzball construction. If we try to make a fuzzball solution using incorrect numbers – for example a wrong value for the tension of the string – then the resulting fuzzball will exhibit pathologies like naked singularities or closed timelike curves. But with the uniquely predicted properties of strings and branes, the fuzzballs are regular quantum solutions of the theory, radiating energy in a way that is in exact agreement with the prediction of thermodynamics.

In our discussion below, we will not follow the historical development of string theory. Instead we will start by examining the symmetries of classical gravity theory. We will see how requiring these symmetries at the quantum level forces us to the rigid structure of string theory, with its definite particle properties and interactions. Knowing these details will then set the stage for us to understand the structure of black holes in the theory, and the consequent resolution of the information paradox.

Lecture notes 1

Extra dimensions

We begin by taking a look at the idea of extra dimensions. This idea precedes string theory; in fact it has been the central tool in most proposals that seek to unify the fundamental forces of nature.

1.1 How gravity differs from other interactions

From the earliest days of physics, there has been a desire to unify different phenomena into a unique overall theory. Two of the most basic interactions in nature are the force of gravity and the force of electrostatics. At a classical, nonrelativistic level, these forces appear very similar:

$$\vec{F}_{gravity} = -\frac{Gm_1m_2}{r^2}\,\hat{r} \tag{1.1}$$

$$\vec{F}_{electrostatics} = \frac{k\,q_1\,q_2}{r^2}\,\hat{r} \tag{1.2}$$

Both are inverse square law forces. But gravity is attractive when both masses have their natural positive value, while the electrostatic force is repulsive between like charges. We now know that this difference signals a very deep difference between the two forces. This difference will become manifest when we incorporate relativistic effects, and will be even more significant in the development of the corresponding quantum theories.

The electromagnetic interaction is mediated by a spin 1 particle – the photon – written as A_μ. The single vector index μ corresponds to the index carried by the electromagnetic current J^μ. The interaction is described by a Lagrangian of the form

$$L_{int} \sim \int A_\mu J^\mu \tag{1.3}$$

The index carried by A_μ allows the contraction of indices which makes the Lagrangian density a scalar, as it should be.

Fig.**??** shows the trajectories of two charged particles, each of which can be thought of as defining a 'current'. The interaction (1.3) causes the first particle to emit a photon, and the same interaction causes the second particle to absorb the photon. This exchange of a photon leads to the electromagnetic interaction between the two particles.

The component J^t encodes the charge q of a particle, and the components $\vec{J} = \{J^x, J^y, J^z\}$ encode the flow of this charge. In the nonrelativistic limit, we can think of the particles as being essentially stationary, and then they are described only by their charges q_i. This gives the interaction $F_{electrostatic}$ in (1.2). (The motion of the charges generates magnetic interactions, which are much weaker than the electrostatic one when all speeds are nonrelativistic: $v_i \ll c$.)

The situation would have been similar with gravity, if we could just replace the charges q_i by masses m_i. But it turns out that the gravitational interaction is proportional to the *energy* E rather than the mass m. In the nonrelativistic limit $E \approx m$, and we recover the interaction $F_{gravity}$ in (1.2). But the difference between E and m is clearly visible when we consider the effect of gravity on photons. Astrophysical observations show that light is deflected by gravity; this is seen for example in the phenomena of gravitational lensing. Photons have $m = 0$, but $E \neq 0$, so we see that the gravitational interaction notices E rather than m.

But E is not a scalar; it is the time component of a 4-vector p^ν. So the scalar charge q gets replaced by a vector p^ν. This fact brings in an extra index ν, so that the current J^μ gets replaced by an object $T^{\mu\nu}$ called the 'stress-energy tensor' or the 'energy-momentum tensor'. Correspondingly, the spin 1 photon A_μ gets replaced by by a spin 2 graviton $h_{\mu\nu}$. At linear order in $h_{\mu\nu}$ the interaction Lagrangian has the form

$$L_{int} = \frac{1}{16\pi G} \int h_{\mu\nu} T^{\mu\nu} \tag{1.4}$$

The component $T^{\mu\nu}$ of the energy-momentum tensor describes the flow of the ν component of the 4-momentum in the spacetime direction μ. One can show that the energy-momentum tensor is symmetric

$$T^{\mu\nu} = T^{\nu\mu} \tag{1.5}$$

Thus we can take $h_{\mu\nu}$ to be symmetric as well. This accords with the fact that the graviton can be thought of as the perturbation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.6}$$

since the metric is naturally symmetric by its definition. The relation (1.6) relates the description of gravity through the exchange of gravitons to our earlier discussion of gravity as a curvature of spacetime.

1.2 Unifying gravity and electromagnetism

Is there any way to unify the electromagnetic theory, which is described by a spin 1 field A_μ, with the gravitational theory which is described by a spin 2 field $h_{\mu\nu}$?

Soon after the advent of general relativity, Kaluza proposed an ingenious way to effect such a unification. Our usual 3+1 dimensional spacetime is described by coordinates x^0, x^1, x^2, x^3. But suppose there is an additional space direction, which we will call x^5. We use indices $\mu, \nu, \ldots$ to range over the usual dimensions $0, 1, 2, 3$ and the indices $A, B, \ldots$ to range over all dimensions $0, 1, 2, 3, 5$. The unperturbed metric then has the form

$$\eta_{AB} = \mathrm{diag}\{-1, 1, 1, 1, 1\} \tag{1.7}$$

A small perturbation to this 4+1 dimensional metric will have the form

$$g_{AB} = \eta_{AB} + h_{AB} \tag{1.8}$$

The information in this metric can be decomposed into the following parts:

(i) The components

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.9}$$

with $\mu, \nu = 0, 1, 2, 3$ are correct in number to describe the metric in 3+1 dimensions.

(ii) The components

$$g_{5\mu} = g_{\mu 5} \equiv A_\mu \tag{1.10}$$

are correct in number to describe the electromagnetic field in 3+1 dimensions.

(iii) The component

$$g_{55} \equiv 1 + C \tag{1.11}$$

describes a scalar field C in 3+1 dimensions.

Thus we seem to get a beautiful unification of gravity and electromagnetism. In the process we get a scalar 'matter' field C as well. Let us see where this line of thought takes us.

1.2.1 Compactification of extra dimensions

A natural question at this stage is: if there is a fifth dimension x^5, then why don't we see it, the same way we see the other space directions x^1, x^2, x^3?

The simplest answer is to assume that the direction x^5 is *small.* Instead of taking the direction x^5 to be an infinite line, we can take x^5 to be a circle of radius R. If R is very small, then our experiments of today cannot resolve this circle, and we do not see it as a new direction. But we still get new components of the metric like $g_{5\mu} = A_\mu$ and $g_{55} = 1 + C$, so we do get the electromagnetic field and the scalar matter field. Since the direction x^5 is now finite in extent (rather than infinite), we call it a 'compact' direction. We say that the full 4+1 dimensional theory has been 'compactified' on a circle S^1 down to 3+1 dimensions.

This compactification solves another issue as well. We want the fields $h_{\mu\nu}, A_\mu, C$ as fields on 3+1 dimensional spacetime. Thus they should be functions of x^0, x^1, x^2, x^3. but not depend on x^5. How can such a requirement be natural?

In the classical theory, making x^5 compact does not imply that the g_{AB} are independent of x^5; all that is required is that the functions have the correct periodicity:

$$g_{AB}(x^5 + 2\pi R) = g_{AB}(x^5) \tag{1.12}$$

But now consider the quantum theory. For convenience, we restrict attention to the single direction x^5. A particle is now described by a wavefunction ψ, which must have the above periodicity

$$\psi(x^5 + 2\pi R) = \psi(x^5) \tag{1.13}$$

The allowed wavefunctions then have the form

$$\psi_n = e^{in\frac{x^5}{R}}, \quad n = 0, \pm 1, \pm 2, \ldots\ldots \tag{1.14}$$

The momentum p_n of such a wavefunction is given by the relation

$$\hat{p}\psi_n = -i\frac{\partial}{\partial x^5}\psi_n = \frac{n}{R}\psi_n \equiv p_n\psi_n \tag{1.15}$$

giving

$$p_n = \frac{n}{R} \tag{1.16}$$

A quantum wavefunction describes the state of some particle in the theory. Suppose this particle had a mass m. Then the momentum (1.16) would imply an energy

$$E_n = \sqrt{p_n^2 + m^2} = \sqrt{\frac{n^2}{R^2} + m^2} \geq \frac{|n|}{R} \tag{1.17}$$

Suppose R is much smaller than the wavelengths λ that we encounter in the lab. Then E_n will be larger than the energies that we can access in our experiments, except for the case $n = 0$. But $n = 0$ corresponds to the wavefunction $\psi_0 = 1$, so this wavefunction is independent of x^5. Thus our requirement that g_{AB} do not depend on x^5 is automatically satisfied when we work at energy scales $E \ll 1/R$.

1.3 The scalar C

So far we have written $G_{55} = 1 + C$ and assumed that C was small. Let us now allow G_{55} to be arbitrary. Note that G_{55} is a positive number, since it gives the length squared of a spacelike direction. We can incorporate this positivity by writing

$$G_{55} = e^C \tag{1.18}$$

For small C, we recover our earlier approximation $G_{55} + 1 + C$.

Let us assume for simplicity that

$$g_{5\mu} = g_{\mu 5} = 0 \tag{1.19}$$

Then the metric has the form

$$g_{AB} = \begin{pmatrix} \begin{pmatrix} & & \\ & g_{\mu\nu} & \\ & & \end{pmatrix} & 0 \\ 0 & g_{55} \end{pmatrix} \tag{1.20}$$

While the form of the metric puts $g_{\mu\nu}$ and g_{55} into different blocks, these two parts of the metric do not actually decouple in the dynamics. This dynamics is governed by the Einstein action

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g_5} R_5 \tag{1.21}$$

where g_5 is the determinant of the 5-dimensional metric g_{AB} and R_5 is the curvature scalar this full metric. We see that

$$\sqrt{-g_5} = \sqrt{-g_4} e^{\frac{C}{2}} \tag{1.22}$$

But the curvature scalar $R_5$is a given by a very nonlinear expression in terms of the components g_{AB}, and R_5 does not separate into an Einstein Lagrangian R_4 describing the dynamics of the components $g_{\mu\nu}$ and an action for the scalar C.

Interestingly, though, a slight redefinition of variables *does* effect such a separation. Define for the directions x^0, x^1, x^2, x^3 a 'rescaled' metric

$$g^E_{\mu\nu} = e^C g_{\mu\nu} \tag{1.23}$$

We can perform some algebraic manipulations to change variables from $g_{\mu\nu}$ to $g^E_{\mu\nu}$ in the action S; we can also do an integration by parts to put the resulting action in standard form. These steps are detailed in Appendix **??**. We find after these steps

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g^E_4} \left(R^E_4 - \frac{1}{2} \partial_\mu C \partial^\mu C \right) \tag{1.24}$$

Here g^E_4 is the determinant of the 4-d metric $g^E_{\mu\nu}$, the curvature scalar R^E_4 is computed using this same 4-d metric, and the indices in $\partial_\mu C \partial^\mu C$ are raised with this metric as well. Since the gravity part of the action has taken the standard form of the 4-d Einstein metric with this choice of variables, we call the rescaled metric (1.21) the *Einstein metric* of the 4-d theory which is obtained by dimensional reduction from 5-d.

1.4 Rescaled metrics

The idea of defining rescaled metrics will be very useful in string theory. So let us pause for a moment to consider this idea in more generality.

When we have a scalar present in our theory, a natural question arises about the definition of the metric. The metric tensor $g_{\mu\nu}$ is a symmetric tensor with two indices. But so is

$$\tilde{g}_{\mu\nu} = e^{\alpha C} g_{\mu\nu} \tag{1.25}$$

for any choice of α. Can we not think of $\tilde{g}_{\mu\nu}$ as the metric instead of $g_{\mu\nu}$?

We have seen that one use of defining a rescaled metric like (1.23) is in the context of dimensional reduction: the rescaling allows us to decouple the action for the scalar C from the metric of the remaining directions. But this is not the only use of rescaled metrics. In our overview of general relativity, we had found two different equations involving the metric:

(a) Given a spacetime with metric $g_{\mu\nu}$, the paths of particles on this spacetime were given by the geodesics of this metric; i.e., the paths with extremal length when length was measured using this metric.

(b) Given sources carrying energy momentum $T_{\mu\nu}$ on our spacetime, the metric was determined by extremizing the Einstein action

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}R \tag{1.26}$$

where we have set the spacetime dimension to be a general integer D.

In string theory, there are many different objects – gravitons, strings, branes etc. Consider any one object, say the string. Then it may be that the motion of this string in spacetime is most conveniently described not by using the metric $g_{\mu\nu}$, but by using a scaled metric (1.25), with an appropriate choice of α. This scaled metric will then be called the 'string metric'.

But if we use this string metric , then the action does not have the simple Einstein form. Thus we often have a choice: we can choose the rescaling (1.25) to make the motion of a chosen object (like a string) look simple, or we can choose the rescaling to make the gravity action simple. What we will now do, however, is focus on a third aspect: with a certain choice of rescaling, we can exhibit important symmetries of the gravity action.

Lecture notes 2

T-duality

In this chapter we observe some very interesting symmetries of solutions of general relativity, in the situation where we have one or more extra dimensions. These symmetries are symmetries of the classical Einstein equations. If we require them to be symmetries of the full quantum theory, then we find a unique structure emerges for this quantum theory; this unique theory of quantum gravity is string theory.

2.1 The symmetry with one scalar

Consider again the action (1.24):

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g_4^E} \left(R_4^E - \frac{1}{2} \partial_\mu C \partial^\mu C \right) \tag{2.1}$$

This action has an obvious symmetry

$$\begin{aligned} C &\rightarrow -C \\ g^E_{\mu\nu} &\rightarrow g^E_{\mu\nu} \end{aligned} \tag{2.2}$$

Note that C appears in the metric in the form $g_{55} = e^C$, and g_{55} itself appears in a very nonlinear way in the 5-d Einstain action. Thus the symmetry (2.2) is a very intersting one. When placed in the appropriate context in string theory, this symmetry will be called *S-duality*.

At the level of the action (2.1) this is a symmetry at the classical level. But it is tempting to require that this symmetry should persist in some way to a symmetry of the full *quantum* theory. It is requirements like this that will lead us to a very rigid and elegant structure to our quantum gravity theory, so that we will arrive at string theory.

Keeping string theory in mind, let us note the analogue of (2.1) for an arbitray number of spacetime dimensions:

(i) Start with $D+1$ dimensional spacetime, where the dynamics is given by the Einstein action

$$S = \frac{1}{16\pi G} \int d^{D+1}x \sqrt{-g_{D+1}} R_{D+1} \tag{2.3}$$

(ii) Compactify the direction x^{D+1} on a circle

$$0 \leq x^{D+1} < B \tag{2.4}$$

with

$$g_{D+1,D+1} = e^C \tag{2.5}$$

We will assume that the metric components do not depend on x^{D+1}. For simplicity, we also assume that $g_{D+1,\mu} = g_{\mu,D+1} = 0$, for $\mu = 0, 1, \ldots D-1$.

(iii) We define

$$g^E_{\mu\nu} = e^{\frac{1}{D-2}C} \tag{2.6}$$

After the steps detailed in Appendix **??**, we find

$$S = \frac{B}{16\pi G}\int d^D x \sqrt{g_E^{(D)}}\left(R_E^{(D)} - \frac{(D-1)}{4(D-2)} C_{,c}C^{,c}\right) \tag{2.7}$$

which again has the symmetry (2.2).

2.2 The symmetry with two scalars

Let us see if we can get more symmetries of this kind. Extending the idea above, we can compactify *two* directions. This will yield two scalars $C, \tilde{C}$, which will describe the sizes of these two compact circles. We can then look for a linear map on $C, \tilde{C}$ that will leave the overall action unchnaged.

We proceed in the following steps:

(i) Start with $D+1$ dimensional spacetime, where the dynamics is given by the Einstein action

$$S = \frac{1}{16\pi G}\int d^{D+1}x \sqrt{-g_{D+1}} R_{D+1} \tag{2.8}$$

(ii) Compactify the direction x^{D+1} on a circle

$$0 \leq x^{D+1} < 2\pi R \tag{2.9}$$

with

$$g_{D+1,D+1} = e^C \tag{2.10}$$

We will assume that the metric components do not depend on x^{D+1}. For simplicity, we also assume that $g_{D+1,\mu} = g_{\mu,D+1} = 0$, for $\mu = 0, 1, \ldots D-1$.

(iii) The metric in the remaining D dimensions is $g_{\mu\nu}$, with $\mu, \nu = 0, 1, \ldots D-1$. Define a rescaled metric

$$g^S_{\mu\nu} = e^{\alpha C} g_{\mu\nu} \tag{2.11}$$

where the constant α will be determined by requirements that we will place below.

(iv) Now compactify a second direction x^{D-1} on a circle

$$0 \le x^{D-1} < A \tag{2.12}$$

with

$$g^S_{D-1,D-1} = e^{\tilde{C}} \tag{2.13}$$

Again, we assume that the metric components do not depend on x^{D-1}. For simplicity we also assume that $g^S_{D-1,i} = g^S_{i,D-1} = 0$, for $i = 0, 1, \ldots D-2$.

Note that A is a quantity with units of length, but it is not the actual length of the x^{D-1} circle. The physical length of this circle involves the metric. If we use the string metric $g^S_{\mu\nu}$ to measure lengths, then the length of this circle is

$$\tilde{L} = e^{\frac{\tilde{C}}{2}} A \tag{2.14}$$

Thus we have two scalars C and $\tilde{C}$, as well as the metric g_{ij} describing the remaining $D-1$ directions, with $i, j = 0, 1, \ldots D-2$.

Now we look for a symmetry similar to (2.2):

$$\begin{aligned} \tilde{C} &\to a\,\tilde{C} + b\,C \\ C &\to c\,\tilde{C} + d\,C \\ g^S_{ij} &\to g^S_{ij} \end{aligned} \tag{2.15}$$

That is, we look for choices of the 5 parameters α, a, b, c, d such that under the map (2.15) the action (2.8) remains invariant.

The relevant algebra is given in Appendix **??**. One finds that the following choice of parameters indeed gives a symmetry:

$$\alpha = \frac{1 \pm \sqrt{D-1}}{D-2}, \quad a = -1, \quad b = 0, \quad c = \mp\frac{2}{\sqrt{D-1}}, \quad d = 1 \tag{2.16}$$

Let us work with the upper sign; the lower sign will be related to this choice by a simple transformation. The symmetry is then

$$\begin{aligned} \tilde{C} &\to -\tilde{C} \\ C &\to C - \frac{2}{\sqrt{D-1}}\tilde{C} \\ g^S_{ij} &\to g^S_{ij} \end{aligned} \tag{2.17}$$

where the 'string metric' $g^S_{\mu\nu}$ is defined by

$$g^S_{\mu\nu} = e^{(\frac{1+\sqrt{D-1}}{D-2})C} g_{\mu\nu} \tag{2.18}$$

Placed in its appropriate context in string theory, this symmetry will be called *T-duality*. Let us look at the nature of this symmetry more closely.

2.3 Making T-duality a symmetry of the quantum theory

The T-duality symmetry (2.17) maps $\tilde{C}$ to $-\tilde{C}$. The new length of the x^{D-1} circle is

$$\tilde{L}' = e^{-\frac{\tilde{C}}{2}} A \; = \; \frac{A^2}{\tilde{L}} \tag{2.19}$$

where the constant A^2 supplies a quantity with the correct units to allow the inversion of the length $\tilde{L}$. Note that the quantity C changes as well under the duality (2.17). We will return to this change later, and focus on (2.19) for the moment. We depict the inversion of $\tilde{L}$ in fig.**??**; where the x^{D-1} circle was large, it becomes small, and where it was small, it becomes large. Given the very nonlinear nature of the gravity action (2.8), it is interesting that a solution of the form fig.**??**(a) can be altered to yield another solution of the form fig.**??**(b).

The symmetry (2.19) is a symmetry of the classical action (2.8). But given its simplicity and elegance, one may want to have it as a symmetry of the full quantum theory of gravity. Is this possible?

Having a symmetry that relates the spacetimes of fig.**??**(a) and (b) means that by *no* experiment should we be able to distinguish which of these two manifolds we have. But using quantum mechanics, we can easily devise an experiment that will differentiate these two manifolds. Consider the marked circle in fig.**??**. In case (a), this circle has length $\tilde{L}$. Any theory of quantum gravity contains one massless particle: the graviton, which arises from the quantization of the gravitational fluctuations $h_{\mu\nu}$. Suppose we look at the wavefunctions that we can have for this graviton on a circle of length $\tilde{L}$. By (**??**) the energy spectrum of such graviton state is given by

$$E_n = \frac{2\pi|n|}{\tilde{L}} \tag{2.20}$$

Thus the spacing between energy levels is $\Delta E = 2\pi/\tilde{L}$.

But if we make the map (2.19), then the length of our circle changes, and the energy spectrum on of the graviton becomes

$$E_n = \frac{2\pi n}{\frac{A^2}{\tilde{L}}} = \frac{2\pi n\tilde{L}}{A^2} \tag{2.21}$$

Now the spacing between energy levels is $\Delta E = 2\pi\tilde{L}/A^2$.

The spacing between energy levels is a measurable quantity in quantum theory. So we can distinguish between the cases (a) and (b) in fig.**??** once we include quantum mechanics, even though these two cases were mapped into each other by an exact symmetry in the classical theory.

Suppose however that we wish to have our T-duality symmetry persist at the quantum level as well. Can we modify the theory in any simple way to achieve this?

In case (a), the energy spacing (2.20) decreases as $\tilde{L}$ is increased, with the falloff $\sim 1/\tilde{L}$. This is natural for wavemodes of a massless particle; longer $\tilde{L}$ means longer wavelengths, and longer wavelengths mean smaller energy. By contrast the energy spacing in (2.21) *grows* linearly with $\tilde{L}$. Is there any object whose energy grows with $\tilde{L}$?

Consider an elastic band. Let the band have tension T; i.e., the energy required to stretch the band by length Δl is

$$\Delta E = T\Delta l \tag{2.22}$$

We assume that the band has a relaxed length of zero, and that T remains the same, regardless of how much we stretch the band.

Now suppose we wrap such a band around the circle in fig.**??**(a). The length of this circle is $\tilde{L}$, so the energy of this wrapped band will be

$$E = T\tilde{L} \tag{2.23}$$

We can wrap the band m times around our circle before gluing its ends together; then the energy will be

$$E = |m|T\tilde{L} \tag{2.24}$$

Here positive values of m correspond to winding one way around the circle, and negative values to the opposite way of winding around the circle.

The energy spectrum (2.24) looks a lot like (2.21), the spectrum we would get after the duality map. To make use of this observation, we make the following modification to our theory of gravity:

(i) We have as usual the graviton in our theory. Wavefunctions of the graviton on a circle of length $\tilde{L}$ give the energy levels

$$E_n = \frac{2\pi|n|}{\tilde{L}} \tag{2.25}$$

(ii) We also assume that there is a string in our theory. This string behaves like an elastic band with relaxed length zero and a constant tension T. Wrapping this string m times around the circle of length $\tilde{L}$ gives the energy levels

$$E_m = |m|T\tilde{L} \tag{2.26}$$

(iii) Thus the overall spectrum of excitations in the theory with a compact circle of length $\tilde{L}$ is

$$E_{mn} = \frac{2\pi|n|}{\tilde{L}} + |m|T\tilde{L} \tag{2.27}$$

(iv) Now suppose we make a T-duality transformation which changes the length of the circle as follows:

$$\tilde{L} \;\rightarrow\; \tilde{L}' \;=\; \frac{A^2}{\tilde{L}} \tag{2.28}$$

Then the graviton modes will give the energy levels

$$E'_{n'} = \frac{2\pi|n'|}{\tilde{L}'} \tag{2.29}$$

and the winding modes of the string will give

$$E'_{m'} = |m'|T\tilde{L}' \tag{2.30}$$

Thus the overall spectrum of excitations will be

$$E_{n'm'} = \frac{2\pi|n'|}{\tilde{L}'} + |m'|T\tilde{L}' = \frac{2\pi|n'|\tilde{L}}{A^2} + \frac{|m'|TA^2}{\tilde{L}} \tag{2.31}$$

(v) If the T-duality map (2.28) is to be a symmetry of the quantum theory, then the excitation spectrum (2.27) should be the same as the excitation spectrum (2.31). This can be achieved if we set

$$\begin{aligned} m' &= n \\ n' &= m \\ T &= \frac{2\pi}{A^2} \end{aligned} \tag{2.32}$$

In other words, the overall spectrum of the circle with length $\tilde{L}$ is now the same as the overall circle with length $\frac{A^2}{\tilde{L}}$. When we change the length of the circle, we must also interchange the 'momentum modes' (2.25) with the 'winding modes' (2.26). The dynamics of a quantum system is completely specified by its energy levels, so we see that the circe of length $\tilde{L}$ cannot be distinguished from the circle with length $\frac{A^2}{\tilde{L}}$. Thus introducing the string in our theory saves the T-duality symmetry of the classical theory from breaking down at the quantum level.

We see that the quantity A has a very fundamental role to play in the definition of T-duality. It is conventional to write

$$T = \frac{2\pi}{A^2} = \frac{1}{2\pi\alpha'} \tag{2.33}$$

where

$$\alpha' = \left(\frac{A}{2\pi}\right)^2 \tag{2.34}$$

is a quantity with units of length squared. We also write

$$\alpha' = l_s^2 \tag{2.35}$$

and call l_s the string length. The reason for this nomenclature is the following. We have assumed that the string is like an elastic band, with relaxed length zero. But quantum fluctuations cause the string to attain a minimal nonzero length, which as we will see is $\sim l_s$. Thus we can think of l_s as setting the length scale where string theory becomes relevant.

Lecture notes 3

Exploring the string

We have seen that classical gravity has a symmetry called T-duality. Requiring that the quantum theory possess this symmetry implies the existence of a string in the theory. Let us explore the properties of this string.

3.1 The action for the string

The configuration where a string is just wrapped on a circle is one of the simplest configurations that we can imagine for the string. But the general configuration looks much more complicated, where different parts of the string stretch and contract, keeping the string in constant movement. How should we describe this dynamics?

A point particle of mass m in general relativity has a very simple action

$$S = -m \int d\tau \tag{3.1}$$

where $d\tau$ is the proper time along an infinitesimal section of the worldline. The string is a 1-dimensional object, and so sweeps out a 2-dimensional 'worldsheet'. It has an equally simple action

$$S = -T \int dA \tag{3.2}$$

Thus extremizing the area of this worldsheet defines the dynamics of the string.

3.2 Why do strings live in 10 dimensions?

We have seen that we can wrap a string around a circle of length L; this generates a winding mode, with mass $m = TL$. What happens if we don't wrap the string on a circle? Then it would seem that the string would collapse to a loop of zero size, with $m = 0$. Can we identify this massless state with the graviton, which is a massless particle that should exist in every quantum theory of gravity?

At a classical level, this certainly looks plausible. But in the quantum theory we have to worry about the fact that the string will always have quantum fluctuations, and we must consider the energy of such fluctuations.

It is easiest to start by looking at the quantum fluctuations of the string around its winding mode. Thus assume that the direction x^{D-1} is compactified to a circle of length $\tilde{L}$, and te string is wrapped once around this circle. Now consider small vibrations of this string in a transverse direction X, where X is one of the directions $X^1, \ldots X^{D-2}$. The action for these vibrations should be obtained from the action (3.2), and turns out to be the same as the action for the vibrations of any string

$$L = \frac{T}{2}\int dtdy[\partial_t X \partial_t X - \partial_y X \partial_y X] \tag{3.3}$$

This is also the Lagrangian of a massless scalar field in 1+1 dimensions. The quantum fluctuations in the ground state of such a field have been well studied, and exhibit something called the Casimir effect. The vacuum energy has the form

$$E_0 = AL - \frac{2\pi}{12L} \tag{3.4}$$

The constant A depends on how we cutoff the high frequency modes in defining our field theory. The contribution of the term AL to E_0 should be absorbed into the definition of the tension T of the string; this can be done because the classical value for E_0 was TL, and so was proportional to L. The contribution $-\frac{1}{12L}$ arises from the fact that the vibration modes of the string are not really continuous, but have a spacing $\Delta E = 2\pi/L$.

Now let us recall our T-duality map (**??**). The energy $2\pi/L$ of the graviton mode was *exact*, in the sense that it included quantum effects. This is because the quantization (**??**) of momentum is an exact relation in quantum mechanics. But then T-duality would require that the energy of the winding mode have the form $E \sim L$, with no added term of the form $\sim 1/L$. What should we do with the Casimir energy term?

The term $-\frac{1}{12L}$ has equal contributions from the left (L) and right (R) modes of vibration

$$-\frac{2\pi}{24L} - \frac{2\pi}{24L} = -\frac{2\pi}{12L} \tag{3.5}$$

Suppose we had d transverse directions of vibration. Then the Casimir energy would be

$$-\frac{2\pi d}{24L} - \frac{2\pi d}{24L} = -\frac{2\pi d}{12L} \tag{3.6}$$

Now suppose we add a left moving vibration, in one of the transverse directions X^i, in the lowest allowed harmonic on the string. The energy of this left vibration would be

$$E_L = \frac{2\pi}{L} \tag{3.7}$$

and it would be characterized by a 'polarization' direction i, with $i = 1, \ldots d$. We similarly add a right vibration, with

$$E_R = \frac{2\pi}{L} \tag{3.8}$$

and a polarization direction j. The total energy of the string is then

$$E = TL - \frac{2\pi d}{24L} - \frac{2\pi d}{24L} + \frac{2\pi}{L} + \frac{2\pi}{L} \tag{3.9}$$

We see that we can cancel the terms with contribution $E \sim 1/L$ by taking

$$d = 24 \tag{3.10}$$

That is, we should have 24 space directions transverse to the string. Including the direction along the string y and the time t, we see that T-duality works if spacetime has $25 + 1$ dimensions.

Making T-duality work has brought in two indices i, j that arise as polarizations on the string. We will now see that these indices are exactly what we need for the T-duality map to make sense. Recall that we had obtained the momentum carrying mode P from a graviton that was moving in the y direction. But a graviton has the form h_{ij}, where i, j are two directions transverse to the direction of motion of the graviton. Thus the momentum mode P also had two indices with the range $i, j = 1, \ldots d$. To summarize, the T-duality map works as follows, in spacetime with dimension $25 + 1$:

(i) The momentum mode P is present in any theory of quantized gravity. We have a massless graviton h_{ij}, moving at the speed of light along the compact direction y, carrying energy and momentum

$$E = P = \frac{2\pi}{L} \tag{3.11}$$

The polarization indices are in the 24 directions transverse to the direction y.

(ii) The duality map changes the length L as

$$L' = \frac{2\pi}{T}\frac{1}{L} \tag{3.12}$$

(iii) The momentum carrying graviton is changed to a string wrapped along the direction y. This string carries a left and a right excitation, with polarizations i, j respectively. With these excitations, the energy of the string is

$$E = TL' = \frac{2\pi}{L} \tag{3.13}$$

where the energy of the vibrations has cancelled the negative Casimir energy. We thus get an agreement under our T-duality map of the energies (3.11) and (3.12), as well as an agreement of the polarizations.

There are two further observations that we need to make:

(a) Consider the string that is *not* wrapped on any circle; classically the lowest allowed energy state would then have $E = 0$. Quantum fluctuations however again bring in a negative Casimir energy. This time we do not have the energy scale $1/L$ that appeared in relations like (3.4). The only energy scale we have is the one set by the tension $T = \frac{1}{2\pi\alpha'}$ of the string. We find

$$m^2 = 8\pi T[-\frac{d}{24} + N] \tag{3.14}$$

where N is the number of oscillators we add for each of the left and right moving vibrations. We again find that the number of transverse dimensions should be 24, and that we should add one left and one right moving vibration. This gives a massless particle ($m^2 = 0$), with two transverse polarization directions i, j. This massless particle can therefore be identified with the graviton. We have therefore found that bosonic string theory lives in a total of $24 + 2 = 26$ spacetime dimensions.

(b) One may choose to *not* add any left or right vibrations to the string. In that case (3.14) gives, with $d = 24$

$$m^2 = -8\pi T \tag{3.15}$$

Such a particle is called a tachyon. It is sometimes said that tachyons travel faster than the speed of light, but this is not the case; the presence of a tachyon signals the fact that the vacuum of the theory is unstable, and that there exists a state with even lower energy – the 'true' vacuum – which will be attained if we add a lot of tachyonic particles. So while we have obtained a massless graviton in our theory by choosing a spacetime dimension $D = 26$, we have a situation where $25 + 1$ dimensional Minkowski spacetime is not the stable vacuum of the theory.

This situation can be remedied if we add fermions to the theory, thus moving from string theory to superstring theory. Before we do this, let us take a brief look at the nature of fermions.

3.3 Fermions

Consider the circle shown in fig.**??**, parametrized by the angular coordinate $0 \le \phi < 2\pi$. We can think of this circle as a circle in the $x - y$ plane. Now consider the wavefunction ψ of a particle living on this circle. We can have the wavefunctions

$$\psi = e^{\frac{i}{\hbar} n\phi} \tag{3.16}$$

where we have temporarily restored the factor $\hbar$ that we have been setting to unity. The angular momentum of this wavefunction is given by

$$J = -i\hbar \frac{\partial}{\partial \phi} = n\hbar \tag{3.17}$$

If we rotate our system around in a complete circle

$$\phi \to \phi + 2\pi \tag{3.18}$$

then the wavefunction (3.16) returns to itself if n is an integer. Thus requiring periodicity of the wavefunction under the transformation (3.18) tells us that the angular momentum is quantized in integral units of $\hbar$.

This angular momentum is termed 'orbital angular momentum'. Elementary particles like electrons have, in addition, an intrinsic angular momentum termed 'spin'. Interestingly, the spin of fermionic particles like electrons and quarks is $\frac{1}{2}\hbar$; thus it is not an integral unit of $\hbar$. Under the transformation (3.18) we then find

$$\psi \to -\psi \tag{3.19}$$

Since we must have returned to the same physical situation after such a rotation, we find that ψ and $-\psi$ should be considered equivalent values of the fermion wavefunction.

Let us now see how this fact will impact string theory when we add in fermions.

3.4 Superstring theory

Consider the string wound on a circle of length L. The transverse vibrations were described by functions $X^i(\sigma), i = 1, \ldots d$ where d is the number of transverse directions. In a supersymmetric theory, we have a fermion for each boson in the theory. Thus we let the string also have fermionic vibrations, described by functions $\psi^i(\sigma), i = 1, \ldots d$.

The bosons were naturally periodic around the σ circle

$$X^i(\sigma) = X^i(\sigma + 2\pi) \tag{3.20}$$

But because of (3.19) we have two possibilities for the fermions:

(i) We let the ψ^i be periodic around the ϕ circle

$$\psi^i(\sigma) = \psi^i(\sigma + 2\pi) \tag{3.21}$$

This is called the Ramond (R) boundary condition. The wavefuntions of the fermions will have the same form as for bosons

$$\psi = e^{in\sigma}, \qquad n \,\text{integer} \tag{3.22}$$

(ii) We let the ψ^i be antiperiodic around the ϕ circle

$$\psi^i(\sigma) = -\psi^i(\sigma + 2\pi) \tag{3.23}$$

This is called the Neveu-Schwarz (NS) boundary condition. The wavefunctions of the fermions will have the form

$$\psi = e^{i(n+\frac{1}{2})}\sigma, \qquad n \,\text{integer} \tag{3.24}$$

The Casimir energy for a boson X on the circle of length L was

$$E = -\frac{2\pi}{24L} - \frac{2\pi}{24L} = -\frac{2\pi}{12L} \tag{3.25}$$

where the two contributions came from the left and right movers. For periodic fermions (3.21) we have

$$E = \frac{2\pi}{24L} + \frac{2\pi}{24L} = \frac{2\pi}{12L} \tag{3.26}$$

For antiperiodic fermions (3.23) we have

$$E = \frac{2\pi}{48L} + \frac{2\pi}{48L} = \frac{2\pi}{24L} \tag{3.27}$$

Now consider d sets of bosons and fermions X^i, ψ^i. Let us focus on the left movers; the right movers give the same numbers. For periodic boundary conditions the Casimir energy would be

$$E = -\frac{2\pi d}{24L} + \frac{2\pi d}{24L} = 0 \tag{3.28}$$

Thus we do not need to add any vibrations to cancel the Casimir energy. For antiperiodic boundary conditions the Casimir energy of the left movers would be

$$E = -\frac{2\pi d}{24L} + \frac{2\pi d}{48L} = -\frac{2\pi d}{48L} \tag{3.29}$$

Let us again try to cancel this by adding a vibration mode. With bosons, the lowest allowed energy of excitation was $2\pi/L$. But with antiperiodic fermions, the vibrations (3.24) give a lowest allowed energy of $\frac{\pi}{L}$, corresponding to the choice $n = 0$. Thus we need to choose d such that

$$-\frac{2\pi d}{48L} + \frac{\pi}{L} = 0 \tag{3.30}$$

which gives

$$d = 8 \tag{3.31}$$

for the number of directions transverse to the string. Adding in the direction along the string and the time direction, we find that spacetime must be $9+1$ dimensional. This is the origin of the statement that superstring theory must live in 10 spacetime dimensions.

Note that the fermion mode ψ^i we added carried a transverse index i. We get a similar index j from the right movers. So just like the bosonic case, the string state where the Casimir energy has been cancelled carries a pair of indices i, j. Extending the analysis to strings that are *not* wrapped on a circle, we again find a massless particle with two indices i, j.

The graviton h_{ij} also has two indices, so we have found a state of the string that can describe the massless graviton. But a closer look reveals that we have actually found more. The graviton is 'transverse traceless and symmetric'. This means the following. Suppose the graviton is moving in the direction X^1. Let us exclude the direction X^1 and the time X^0, and let i, j range over all the other space directions $X^2, \dots X^{D-1}$. Then we have a graviton state is characterized by a symmetric and traceless matrix h_{ij}.

We have indeed found that the indices i, j range over the transverse values $X^2, \dots X^{D-1}$. But we can make any matrix M_{ij} in this transverse space, since we can choose any value of the index i from the left movers and the index j from the right movers. We can find the graviton in this subspace of states M_{ij} by decomposing M_{ij} into three parts

$$M_{ij} = h_{ij} + B_{ij} + \phi \tag{3.32}$$

where

$$\begin{aligned} h_{ij} &= \frac{1}{2}(M_{ij} + M_{ji}) - \frac{1}{D-2}\sum_k M_{kk} \\ B_{ij} &= \frac{1}{2}(M_{ij} - M_{ji}) \\ \phi &= \frac{1}{D-2}\sum_k M_{kk} \end{aligned} \tag{3.33}$$

The part h_{ij} is symmetric and traceless, and gives the graviton. The part B_{ij} is antisymmetric. We will soon see that it gives the gauge field coupling to the string, just as a vector potential A_i gives the gauge field coupling to an electron. The last part ϕ is a scalar. How should we interpret ϕ?

We have seen that if we have an extra compact direction, then the size of this direction can be thought of as describing scalar $\phi(x)$ in the remaining directions. String theory lived in 9+1 dimensions, but of we interpret this scalar ϕ as the size of an extra direction, then we have 10+1 spacetime dimensions. Thus we have two ways of studying our theory:

(i) As string theory in 9+1 dimensions. Here ϕ is a scalar field in the theory, called the dilaton. It has a very important role to play, as it gives the strength of coupling between strings.

(ii) As a theory in 10+1 dimensions, where one of the space directions has been compactified to a circle. This direction is traditionally called x^{11}, and the length of this circle is $2\pi R_{11}$. This 10+1 dimensional theory is called M-theory.

The 10+1 dimesnional M-theory description provides a remarkable perspective on the structure of string theory, giving a simple picture of all the different elementary excitations of the theory. For actual computations, however, the 9+1 dimensional string theory perspective is often more useful. The reason is that M-theory has no small parameters in it. But if we compactify on a circle to get string theory, then we can consider the limit where this circle is small, and then this smallness serves as a useful expansion parameter.

3.4.1 Charged particles

Given that we have discovered an extra circle x^{11} in string theory, we now automatically find new fundamental objects. Consider the 10+1 dimensional gravity theory. We can take the massless graviton of this theory and give it a wavefunction

$$\psi = e^{i\frac{x^{11}}{R_{11}}} \tag{3.34}$$

This graviton has an energy and momentum

$$E = P = \frac{1}{R_{11}} \tag{3.35}$$

From the 9+1 dimensional string theory perspective, this looks like a pointlike object with mass $m = E$. This object is called the D0 brane. The prefix D will be explained shortly, but the number 0 says that this is a pointlike object – an object of 0 spatial dimensions. We could have also chosen the wavefunction

$$\psi = e^{-i\frac{x^{11}}{R_{11}}} \tag{3.36}$$

This graviton has an energy and momentum

$$E = -P = \frac{1}{R_{11}} \tag{3.37}$$

The two choices $P = \pm E$ suggests that we have two particles of equal and opposite charges, and indeed P can be interpreted as a charge of the pointlike particle that we see in the 9+1 dimensional string theory perspective.

But if the D0 brane is a charged particle, then there should be a gauge field A_μ that it should couple to. There is indeed such a gauge field. When we have an extra compact circle x^{11}, then we had seen that we get the field

$$g_{11,\mu} \sim A_\mu \tag{3.38}$$

Investigation of the gravity Lagrangian reveals that this gauge field does couple to the D0 brane exactly as required for the normal coupling of a gauge field to a charged particle.

3.4.2 BPS particles

If P gives the charge q of the D0 brane, and E gives the mass m, then we find that

$$m = |q| \tag{3.39}$$

that is, the strength of the charge q equals the mass. We will see that this is a basic property of all the elementary objects in string theory. This property will play a very central role in our study of black holes. A particle satisfying the relation (3.39) is called a BPS particle. This term refers to the Bogolmoyi-Prasad-Sommerfeld bound first noted for monopoles, where it was found that all states in the theory had $m \geq |q|$.

Usually we think of mass and charge being measured in different units, so let us see what (3.39) means in physical terms. Consider two BPS particles with the same charge. Place one at at the location $\vec{x}$ and another at the location $\vec{x}'$. There are two forces between these particles:

(i) The force of gravitational attraction. We take the particles to to be well separated, and then we can use the Newtonian expression (**??**) for this gravitational attraction.

(ii) The force of electrostatic repulsion, which arises from the fact that each particle carries a charge. If the particles are taken to be well separated, then we can use the classical formula (1.2) for this electrostatic repulsion.

For a BPS particle, these two forces exactly cancel; i.e., there is no net force between two copies of the same particle.

Interestingly, the BPS relation (3.39) arises in three different ways:

(a) From the natural charge and mass obtained by dimensional reduction of massless particles, as in (3.39).

(b) In supersymmetric theories we will see that

$$M \geq |Q| \tag{3.40}$$

for all states of the theory. BPS particles have $M = |Q|$, and so saturate the bound. The supersymmetry is characterized by a set of characterized by a set of supersymmetry operators $\hat{\mathcal{Q}}_\alpha$. BPS particles are described by states $|\psi\rangle$ satisfying

$$\hat{\mathcal{Q}}|\psi\rangle = 0 \tag{3.41}$$

where $\hat{\mathcal{Q}}$ is a linear combination of the $\hat{\mathcal{Q}}_\alpha$.

(c) Black holes have a mass M, but we can also allow them to have a charge Q. It turns out that when we write the metric produced by an object of mass M and charge Q, then we get a good solution only for

$$M \geq |Q| \tag{3.42}$$

The limiting case

$$M = |Q| \tag{3.43}$$

is called the 'extremal' hole. If we try to write a solution for $M < |Q|$, then we find that the singularity of the black hole is *outside* the horizon. Such a singularity is called a naked singularity, and it is unclear how to define physics in the presence of such singularities. Extremal black holes have a Hawking temperature $T_H = 0$. This means that a hole cannot radiate neutral particles and reach a situation with $M < |Q|$, so the bound (3.42) is preserved during the natural evolution of the hole.

In supersymmetric theories, we will see that (b) can be derived from (a). That (c) is true has a very interesting implication. One would like to think that black holes are just another combination of elementary objects in our quantum gravity theory, and their behavior is just like that of any other bound state in the theory. But we have seen that this expectation runs into trouble: we cannot find the states which should account for the entropy of the hole, and the evaporation of the hole leads to difficulties with unitarity. The inequality (3.42) satisfied by classical black holes provides some indication that a understanding of the black hole as a normal bound state may be possible. Since all states in a supersymmetric theory satisfy (3.40), it is reassuring that black holes do not violate this inequality. We will find that the simplest black holes in string theory are the extremal ones with $M = |Q|$, and that they can indeed be understood a bound states of the elementary constituents of the theory.

3.5 The scalar ϕ as a coupling

We have seen that quantizing the string gives rise to a massless scalar ϕ. We can interpret this scalar as encoding the size of an extra direction x_{11}, giving a 10+1 dimensional theory called M-theory. We will find that this scalar plays a very fundamental role in the 9+1 dimensional string theory: it gives the coupling constant g of the theory. Let us make this relation precise.

The 11-D M theory has the gravitational action

$$S = \frac{1}{16\pi G}\int d^{11}x\sqrt{-g_{11}}R_{11} \tag{3.44}$$

Following our earlier notation, let us write

$$g_{11,11} = e^C \tag{3.45}$$

and for the moment assume

$$g_{11,\mu} = 0 \tag{3.46}$$

Let us choose the compact circle x_{11} to have a coordinate range

$$0 \le x_{11} < 2\pi R_{11} \tag{3.47}$$

Then the dimensionally reduced 9+1 dimensional action has the form

$$S = \frac{2\pi R_{11}}{16\pi G}\int d^{10}x\sqrt{-g_{10}}e^{\frac{C}{2}}[R_{10} + \ldots] \tag{3.48}$$

Recall that we defined a string metric

$$g^S_{\mu\nu} = e^{\frac{C}{2}}g_{\mu\nu} \tag{3.49}$$

in terms of which T-duality could be expressed in a simple way. In terms of this metric, we find

$$\sqrt{-g_{10}} = e^{-\frac{5C}{2}}\sqrt{-g^S_{10}}, \quad R_{10} = e^{-C}[R^S_{10} + \ldots] \tag{3.50}$$

so that we have

$$S = \frac{2\pi R_{11}}{16\pi G} \int d^{10}x e^{-\frac{3C}{2}} \sqrt{-g_{10}^S}[R_{10}^S + \dots] \tag{3.51}$$

Now we recall that in an interacting theory like electromagnetism, the coupling constant e shows up in the action as follows

$$S = -\frac{1}{4e^2} \int d^4x F^2 \tag{3.52}$$

Thus G acts like the coupling constant squared of the gravitational theory. But since we also have the factor $e^{-\frac{3C}{2}}$ in the action, we see that the effective coupling is

$$g^2 \sim G e^{\frac{3C}{2}} \tag{3.53}$$

A larger value for C implies a larger size for the circle x_{11}. Thus the size of this extra circle governs the coupling constant of string theory. Note that C can be different at different points, so that this coupling is a *field*, rather than a fixed constant. We will find this fact to be very useful in our analysis of black holes: we can imagine the coupling to be weak or strong, and thus relate computations at weak coupling (where we do not expect black holes) to computations at strong coupling (where we do expect black holes).

Since g will play a central role in our theory, we define a field ϕ through

$$g = e^{\phi} \tag{3.54}$$

From (3.53) we see that

$$C = \frac{4\phi}{3} \tag{3.55}$$

Note that we have not yet obtained a complete definition of g. We have related ϕ to C through (3.55), and C was related to the size of the compact direction. But we have not yet specified R_{11}, which governs the coordinate range for x_{11} through (3.47). Changing this coordinate range will change C, which will change ϕ and therefore g. How should we choose R_{11}?

Recall that to see T-duality, we compactify a direction x^9 as

$$0 \le x_9 \le 2\pi R_9 \tag{3.56}$$

and use the string metric to measure distances. The T-duality map is then

$$R_9' = \frac{l_s^2}{R_9} \tag{3.57}$$

Thus the value

$$R_9 = l_s \tag{3.58}$$

gives a 'self-dual' point; a circle of this length maps back to itself under T-duality.

This was the perspective of string theory. But now consider the perspective of M-theory, which has an additional circle x_{11}. In the above setting, we have two compact circles, x_9 and x_{11}. Now we can perform two different kinds of operations:

(a) Consider x_{11} to be the extra circle of M-theory. This gives a 9+1 dimensional string theory, where another direction x_9 has been compactified.

(b) Consider x_9 to be the extra circle of M-theory. This also gives a 9+1 dimensional string theory, where another direction x_{11} has been compactified.

These two different ways of viewing the theory is called the $9-11$ flip. We will soon see its significance. For now, we will fix the normalization of g as follows. Suppose we set the length of the x_9 circle at its self-dual value

$$L_9 = 2\pi R_9 = 2\pi l_s \tag{3.59}$$

Then we require that the value $g = 1$ describe the situation where the x_9 and x_{11} circles have the same length.

Let us see what we get by imposing this requirement. With the string metric (3.49), the metric in the x_{11} direction is

$$g^S_{11,11} = e^{\frac{C}{2}} g_{11,11} = e^{\frac{C}{2}} e^C = e^{\frac{3C}{2}} \tag{3.60}$$

In this string metric, the length of the direction x_{11} is

$$L_{11} = 2\pi R_{11} e^{\frac{3C}{4}} = 2\pi R_{11} e^{\phi} = 2\pi R_{11} g \tag{3.61}$$

When $g = 1$, this is $L_{11} = 2\pi R_{11}$. We set this equal to the length of the x_9 direction at the self dual point $L_9 = 2\pi l_s$. Thus

$$2\pi R_{11} = 2\pi l_s \tag{3.62}$$

which gives

$$R_{11} = l_s \tag{3.63}$$

This is the relation that we were seeking. With this relation, the length of the x_{11} direction, at a general value of g, is from (3.61)

$$L_{11} = 2\pi l_s g \tag{3.64}$$

3.5.1 T-duality

With the relation (3.64), we can write the mass of a D0 brane (**??**) in an interesting form:

$$m_{D0} = \frac{2\pi}{L_{11}} = \frac{1}{g l_s} \tag{3.65}$$

Thus the mass, measured in string units, is given in terms of the coupling constant of the theory. This is interesting; in the usual theory of electroweak interactions, there is no relation between the coupling and the masses of particles. There is one kind of object in field theory, however, whose mass is related to the coupling. This is a soliton, given by a classical solution to the field equations. Given the form of the action (3.52), one finds that the mass of a soliton is $m \sim \frac{1}{e^2}$, where e is the coupling. The elementary particles (i.e. those which are not solitons) have masses that have no powers of e. In the present case we have $m \sim 1/g$, so the power of the coupling is halfway between that of elementary particles and solitions. We will see that this difference is a matter of which metric we choose: with one metric we can make the D-brane look like an elementary particle, while with another metric we can make it look like a soliton.

Let us now recall the T-duality transformation (2.17), where we now set $D = 10$:

$$\begin{aligned} \tilde{C} &\rightarrow -\tilde{C} \\ C &\rightarrow C - \frac{2}{3}\tilde{C} \\ g^S_{ij} &\rightarrow g^S_{ij} \end{aligned} \tag{3.66}$$

The length of the x_9 circle is

$$L_9 = 2\pi R_9 e^{\frac{1}{2}\tilde{C}} \tag{3.67}$$

The first relation tells us that the length of x_9 changes as

$$L'_9 = \frac{(2\pi)^2 l_s^2}{L_9} \tag{3.68}$$

The next relation tells us that

$$g' = e^{\phi'} = e^{\frac{3}{4}C'} = e^{\frac{3}{4}C} e^{-\frac{1}{2}\tilde{C}} = e^{\phi}\frac{l_s}{L} = g\frac{l_s}{L} \tag{3.69}$$

Now we can ask the question: we want T-duality to be a symmetry of our quantum theory. Under the T-duality map, a graviton moving along x_9 became a string wrapped along x_9, and a string wrapped along x_9 became a graviton along x_9. But the D0 brane is another object in the theory? What happens to it when we apply the T-duality transformation?

Since T-duality is a symmetry, and the string length did not change under this duality, the mass of the object should remain

$$m = \frac{1}{g l_s} \tag{3.70}$$

We shoud however write this in terms of the new coupling; thus

$$m = \frac{1}{g' l_s}\frac{l_s}{L} = \frac{1}{g' l_s}\frac{L'}{l_s} \tag{3.71}$$

where in the last step we have written everything in terms of the quantities after the duality. We now see something interesting; the mass is linearly proportional to the new length of the x_9 direction. This tells us that the new object obtained after the duality should be some kind of a string, wrapped along the x_9 direction. We will call this string a D1-brane. The tension of this D1 brane is defined by $m = T_{D1}L'_9$, which gives

$$T_{D1} = \frac{1}{g' l_s^2} \tag{3.72}$$

Thus we find that, measured in string units, the tension of the D1 brane is again the reciprocal of the coupling.

3.6 Fermions

Thus far we have mostly ignored the fermions of our theory. But at this stage in our discussion of T-duality, they play a crucial role. When we do a T-duality, the fermions change their nature in a way we will now describe. The theory we obtained from 11-d M theory by compactfication on a circle is called type IIA string theory, for reasons we will see shortly. After a T-duality in a direction like x_9, the fermions change their nature so that we get type IIB string theory. A second T-duality will bring us back to IIA and so on. To understand the difference between IIA and IIB theories, let us recall what fermions are.

3.6.1 Spinors

A vector V^a is a representation of the rotation group. This group is the set of all transformations that leave the length of the vector unchanged. Tensors are then built by by taking linear combinations of products like V^aW^b, and give other representations of the rotation group.

But there is a group that is simpler than the rotation group; this is the group of *reflections*. A reflection is done across a plane, as shown in fig.**??**(a). A reflection also keeps the length of a vector unchanged. In fig.**??**(b) shows two planes, with an angle $\theta/2$ between them. Reflecting in the first plane and then in the second plane results in a rotation through an angle θ. Thus a reflection is a 'square-root' of a rotation. The set of rotations in all possible planes is the group of reflections. Each plane is described by its normal vector $\hat{n}$.

We would like a representation of the reflection group. That is, we want a set of numbers $\{s_1, \dots s_n\}$ that will be multiplied by a matrix $\Gamma[\hat{n}]$ under reflection in the plane with normal $\hat{n}$. Consider 3-d space x^1, x^2, x^3. Let Γ^i be the matrix for reflection in the plane with normal in the direction x^i. Reflecting twice in the same plane brings a vector back to itself, so we need

$$(\Gamma^i)^2 = I, \quad i = 1, 2, 3 \tag{3.73}$$

Now consider a plane with normal n^i. For reflection in this plane we take the matrix $n^i\Gamma^i$. For $n^i = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Then we have

$$\Gamma = \frac{1}{\sqrt{2}}\Gamma^1 + \frac{1}{\sqrt{2}}\Gamma^2 \tag{3.74}$$

We again need $\Gamma^2 = I$. We find

$$\Gamma^2 = \frac{1}{2}I + \frac{1}{2}I + \frac{1}{2}[\Gamma^1\Gamma^2 + \Gamma^2\Gamma^1] \tag{3.75}$$

Thus we need

$$[\Gamma^1, \Gamma^2]_+ \equiv [\Gamma^1\Gamma^2 + \Gamma^2\Gamma^1] = 0 \tag{3.76}$$

i.e., different Γ^i *anticommute*.

It is easy to find three matrices Γ^i with these properties

$$\Gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.77}$$

The fact that these are 2×2 matrices tells us that the representation is 2-dimensional; i.e., the spinor will have 2 states. These give the two spin states of the electron – up and down. But it is more useful to consider the representation given by the Γ^i in a more abstract way. First we note that Γ^3 can be chosen as

$$\Gamma^3 = -i\Gamma^1\Gamma^2 \tag{3.78}$$

since this satisfies the requirements (3.75) and (3.76). For the remaining Γ^i, we define the 'raising' and 'lowering' combinations

$$\Gamma^+ = \frac{1}{\sqrt{2}}(\Gamma^1 + i\Gamma^2), \quad \Gamma^+ = \frac{1}{\sqrt{2}}(\Gamma^1 + i\Gamma^2) \tag{3.79}$$

which satisfy

$$(\Gamma^+)^2 = 0, \quad (\Gamma^-)^2 = 0, \quad [\Gamma^+, \Gamma^-]_+ = I \tag{3.80}$$

As a starting state of the representation, we consider the 'lowest weight state' $|-\rangle$, defined by

$$\Gamma^-|-\rangle = 0 \tag{3.81}$$

We can generate another state through

$$\Gamma^+|-\rangle \equiv |+\rangle \tag{3.82}$$

but since $(\Gamma^+)^2 = 0$, we cannot raise the state any further. We also have

$$\Gamma^-|+\rangle = \Gamma^-\Gamma^+|-\rangle = [\Gamma^-, \Gamma^+]_+|-\rangle = |-\rangle \tag{3.83}$$

so the raising and lowering operations toggle between the states $|\pm\rangle$. We also find

$$\begin{aligned} \Gamma^3|-\rangle &= -i\Gamma^1\Gamma^2|-\rangle = -\frac{1}{2}(\Gamma^+ + \Gamma^-)(\Gamma^+ - \Gamma^-)|-\rangle = -|-\rangle \\ \Gamma^3|+\rangle &= |+\rangle \end{aligned} \tag{3.84}$$

so no new states arise, and $|\pm\rangle$ form a complete representation.

Rotation through an infinitesimal angle $\delta\theta$ in the $x-y$ plane is given by the operation

$$(\Gamma^1 + \frac{\delta\theta}{2}\Gamma^2)\Gamma^1 = I + i\frac{\delta\theta}{2}\Gamma^3 \approx e^{i\frac{\delta\theta}{2}\sigma_3} \tag{3.85}$$

Composing such infinitesimal rotations we find that under rotation through an angle θ

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \rightarrow e^{i\frac{\theta}{2}\sigma_3}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}} & \\ & e^{-i\frac{\theta}{2}} \end{pmatrix}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \tag{3.86}$$

Thus rotation through $\theta = 2\pi$ does not bring the states $|\pm\rangle$ back to themselves; we need a rotation through $\theta = 4\pi$. Thus particles like the electrons described by such spinors have spin $\frac{1}{2}$, and their wavefunction changes sign under a 2π rotation. We have used this fact in section **??**, when considering the possibility of antiperiodic boundary conditions for fermions.

We can make spinor representations in any dimension in a similar way, so let us take the case of M-theory. The spacetime is 10+1 dimensional, but for simplicity we rotate time as $t \rightarrow -i\tau$ and look at 11 Euclidean dimensions for a start. We have 11 matrices Γ^i, satisfying

$$[\Gamma^i, \Gamma^j]_+ = 2\delta^{ij} \tag{3.87}$$

We can write

$$\Gamma^{11} = -i\Gamma^1\Gamma^2\dots\Gamma^{10} \tag{3.88}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{2}}(\Gamma^1 \pm i\Gamma^2) = \Gamma^{\pm}_{(1)}, \quad \frac{1}{\sqrt{2}}(\Gamma^3 \pm i\Gamma^4) = \Gamma^{\pm}_{(2)}, \quad \frac{1}{\sqrt{2}}(\Gamma^5 \pm i\Gamma^6) = \Gamma^{\pm}_{(3)} \\ &\frac{1}{\sqrt{2}}(\Gamma^7 \pm i\Gamma^8) = \Gamma^{\pm}_{(4)}, \quad \frac{1}{\sqrt{2}}(\Gamma^9 \pm i\Gamma^{10}) = \Gamma^{\pm}_{(5)} \end{aligned} \tag{3.89}$$

The lowest weight state $|0\rangle$ is defined by

$$\Gamma^-_{(a)}|0\rangle = 0 \tag{3.90}$$

We can act on this state by the raising operators $\Gamma^+_{(a)}$. We can apply each raising operator 0 or 1 times, so we have $2^5 = 32$ states in all. For states obtained by applying an even number of raising operators

$$|\psi\rangle_{even}: \quad |0\rangle, \quad \Gamma^+_{(a)}\Gamma^+_{(b)}|0\rangle, \quad \Gamma^+_{(a)}\Gamma^+_{(b)}\Gamma^+_{(c)}\Gamma^+_{(d)}|0\rangle \tag{3.91}$$

we find that

$$\Gamma^{11}|\psi\rangle_{even} = |\psi\rangle_{even} \tag{3.92}$$

while for states obtained by applying an off number of raising operators

$$|\psi\rangle_{odd}: \quad \Gamma^+_{(a)}|0\rangle, \quad \Gamma^+_{(a)}\Gamma^+_{(b)}\Gamma^+_{(c)}|0\rangle, \quad \Gamma^+_{(a)}\Gamma^+_{(b)}\Gamma^+_{(c)}\Gamma^+_{(d)}\Gamma^+_{(e)}|0\rangle \tag{3.93}$$

we find that

$$\Gamma^{11}|\psi\rangle_{odd} = -|\psi\rangle_{odd} \tag{3.94}$$

There are 16 states $|\psi\rangle_{even}$ satisfying (3.92) are called states with positive chirality, and the 16 states $|\psi\rangle_{odd}$ satisfying (3.94) are called states with negative chirality.

Let us now explain why all this is important at the present stage of our discussion of T-duality. The 11-d M-theory has spinors with 32 components. The string theory is however 10 dimensional, so the reflection group only involves matrices $\Gamma^1 \ldots \Gamma^{10}$, but no matrix Γ^{11}. In 10 dimensions it turns out that the spinor representations have only 16 components: we can either take the set $|\psi\rangle_{even}$ or the set $|\psi\rangle_{odd}$. Since our string theory is really rewrite of an 11-d theory, all 32 components of the M theory spinor have to show up, and so we get one 16 dimensional positive chirality representation and 16 dimensional negative chirality representation. The string theory obtained by reducing M-theory on the x^{11} circle is called type IIA string theory; the II stands for the fact that we have found two spinor representations, and the letter A will serve to distinguish this theory from type IIB string theory that we will find presently.

Now suppose we make one direction x_9 compact. We can do a T-duality along x_9. This T-duality takes us from 10-dimensions back to 10-dimensions, but *one of the two* 16 *component spinors flips its chirality.* Since type IIA theory had two spinors of opposite chirality, now we have two spinors of the same chirality.

This is clearly a different theory, and we call it type IIB string theory. We cannot get this theory by dimensional reduction from 11-d; the 11-d theory has one 32 component spinor, and this always breaks up into a pair of opposite chirality spinors in the 10-d reduced theory. So this theory has to be understood only as a 10-d string theory.

The D0 brane we had found was in type IIA string theory; this brane came from looking at the 11-d theory and considering a graviton moving along x^{11}. But the D1 brane we found was obtained by a T-duality. This T-duality brings us to type IIB string theory, so the D1 brane is an object not i the original IIA theory, but in the IIB theory.

The T-duality in x^9 had converted the D0 brane of IIA into a D1 brane wrapped along x^9 in IIB string theory. Now imagine compactifying a direction x^8 in this IIB string theory. We can do a T-duality in x^8, and ask what happens to the D1 brane wrapped along x^9. As in the case of the D0 brane, we argue that the mass of the brane in string units must not change. We are led to a new object, which is now a D2 brane, with tension

$$T_{D2} = \frac{1}{g l_s^2} \tag{3.95}$$

This second T-duality has brought us back to IIA theory, so the D2 brane is an object in IIA string theory. Proceeding in this way we find the objects:

$$\begin{array}{lcl} IIA & : & D0, \quad D2, \quad D4, \quad D6 \\ IIB & : & D1, \quad D3, \quad D5 \end{array} \tag{3.96}$$

we can also find D7, D8 and D9 branes, but these objects are a little different from the other branes. A D7 brane has only 2 transverse space dimensions, and its gravitational potential behaves as $\ln r$ where r is the distance from the brane. Thus its potential does not die off at infinity. In making black holes we will be seeking to get spacetime that is flat at infinity, so we will not use branes higher than D6.

The tensions of these branes is

$$T_{Dp} = \frac{1}{g l_s^{p+1}} \tag{3.97}$$

Lecture notes 4

Electric-magnetic duality

4.1 Gauge fields

A charged particle like the electron couples to the gauge field A_μ describing electromagnetism. String theory will have charged branes which generalize the notion of charged particles, and corresponding gauge fields which generalize A_μ.

Let us first note how a particle with charge q couples to a gauge field A_μ in curved spacetime:

(i) Consider the worldline described the the charged particle in spacetime. Consider an infinitesimal segment of this worldline, which extends from the spacetime point ξ^μ to $\xi^\mu + d\xi^\mu$.

(ii) Let $A_\mu(\xi)$ be the value of the gauge field at the point ξ^μ. Compute the scalar $A_\mu(\xi)d\xi^\mu$.

(iii) The action S_{int} describing the interaction of the charged particle with the gauge field is given by adding these contributions over the worldline

$$S_{int} = q \int A_\mu d\xi^\mu \tag{4.1}$$

Since the charge q is typically quantized in integral units of a basic value q_0, we absorb q_0 into the definition of A_μ. Then particles with unit charge have the interaction

$$S_{int} = \int A_\mu d\xi^\mu \tag{4.2}$$

We will use a similar convention when studying branes below.

Now consider a string. This is a 1-dimensional object, so it sweeps out a 2-dimensional 'world-sheet' in spacetime. We follow the same steps as before:

(i) Consider an infinitesimal area on the worldsheet, located around a point ξ^μ in spacetime. This area element is described by a parallogram, with sides $d\xi^\mu_{(1)}, d\xi^\mu_{(2)}$. Recall that in ordinary 3-d space, the area of two parallogram defined by vectors $\vec{V}, \vec{W}$ lying in the $x-y$ plane is given by the cross product

$$A = \hat{z} \cdot (\vec{V} \times \vec{W}) = V_x W_y - V_y W_x \tag{4.3}$$

In our present case, the infinitesimal area element will be described by

$$d\xi^{\mu}_{(1)}d\xi^{\nu}_{(2)} - d\xi^{\nu}_{(1)}d\xi^{\mu}_{(2)} \tag{4.4}$$

(ii) The gauge field will in this case have two indices: $B_{\mu\nu}(\xi)$. Compute the scalar

$$\frac{1}{2}B_{\mu\nu}(d\xi^{\mu}_{(1)}d\xi^{\nu}_{(2)} - d\xi^{\nu}_{(1)}d\xi^{\mu}_{(2)}) \tag{4.5}$$

Note that we can assume that the gauge field is antisymmetric

$$B_{\mu\nu} = -B_{\nu\mu} \tag{4.6}$$

since the gauge field multiples the antisymmetric tensor defining the area element.

(iii) The interaction of the string with the gauge field is given by integrating over all the area elements on the worldsheet

$$S_{int} = \int \frac{1}{2}B_{\mu\nu}(d\xi^{\mu}_{(1)}d\xi^{\nu}_{(2)} - d\xi^{\nu}_{(1)}d\xi^{\mu}_{(2)}) \equiv \int BdA \tag{4.7}$$

Thus we need a field $B_{\mu\nu}$ to serve as the gauge field coupling to the string. This field should be massless, as all gauge fields are, and should be antisymmetric (eq. 4.6)). But in section **??** we have found a field with just these properties: some of the massless modes of the string produced the graviton $h_{\mu\nu}$, but other have an antisymmetric tensor $B_{\mu\nu}$. This is an example of the beautiful interlocking nature of string theory: each object that is needed in the theory is naturally present in the theory, without having to be added as an optional feature.

We have a similar coupling to gauge fields for the higher dimensional branes of the theory. Consider a p-brane; i.e., a brane extended in p space directions. This brane sweeps out a $p+1$ dimensional worldvolume in spacetime. We follow the same steps again:

(i) Consider an infinitesimal hypercube on the worldvolume, located around a point ξ^μ in spacetime. This hypercube has sides $d\xi^{\mu}_{(1)}, d\xi^{\mu}_{(2)}, \ldots d\xi^{\mu}_{(p+1)}$. It defines a volume element

$$dV : \quad \frac{1}{(p+1)!}[d\xi^{\mu_1}_{(1)}d\xi^{\mu_2}_{(2)}\ldots d\xi^{\mu_{p+1}}_{(p+1)} + \text{permutations}] \tag{4.8}$$

where we add terms with all permutations of the indices $\mu_1, \ldots, \mu_{p+1}$, with sign given by the sign of the permutation.

(ii) The gauge field will have $p+1$ indices: $A_{\mu_1\mu_2\ldots\mu_{p+1}}(\xi)$, and will be antisymmetric under the interchange of any pair of indices. Compute the scalar

$$A_{\mu_1\mu_2\ldots\mu_{p+1}}(\xi)d\xi^{\mu_1}_{(1)}d\xi^{\mu_2}_{(2)}\ldots d\xi^{\mu_{p+1}}_{(p+1)} \tag{4.9}$$

where the antisymmetry of A takes into account the permutations in (4.8).

(iii) The interaction of the string with the gauge field is given by integrating over all the area elements on the worldsheet

$$S_{int} = \int A_{\mu_1\mu_2\ldots\mu_{p+1}}(\xi) d\xi^{\mu_1}_{(1)} d\xi^{\mu_2}_{(2)} \ldots d\xi^{\mu_{p+1}}_{(p+1)} \equiv \int A dV \tag{4.10}$$

Where would we find the required gauge fields $A_{\mu_1\mu_2\ldots\mu_{p+1}}$? Recall that the string had two kinds of states: the NS sector states where the fermions were antiperiodic around the string, and the R sector states where the fermions were periodic. When both left and right moving fermions are in the NS sector, we obtained the fields $h_{\mu\nu}, B_{\mu\nu}, \phi$. It turns out that when both left and right movers are in the R sector, we get exactly the needed gauge fields $A_{\mu_1\mu_2\ldots\mu_{p+1}}$. We can also take the left movers in the NS sector and the right movers in the R sector (or vice versa); this produces fermionic fields, which naturally pair up with the bosonic fields $h_{\mu\nu}, B_{\mu\nu}, \phi, A_{\mu_1\mu_2\ldots\mu_{p+1}}$ as required by supersymmetry.

4.2 Electric-Magnetic duality

Maxwell's theory of electromagnetism has a beautiful symmetry

$$\vec{E} \leftrightarrow \vec{B} \tag{4.11}$$

as long as we have no sources for the fields. But the sources we find in nature violate this symmetry. While we find charged particles that are a source for $\vec{E}$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{4.12}$$

there is no source for $\vec{B}$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4.13}$$

People have long wondered whether there could be magnetic monopoles – sources of charge for $\vec{B}$, so that we would have

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m \tag{4.14}$$

Then we would have an exact electric-magnetic duality symmetry in the theory, and electrically charged particles like electrons would map under this duality to magnetically charged monopoles.

In this section we will see that in string theory we *do* have exact electric-magnetic duality.

4.2.1 The duality in electromagnetism

To understand the structure of the duality, let us first rewrite the electromagnetic case in relativistic form. The basic variable is a gauge field A_a. This gives a field strength

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{4.15}$$

The traditional electromagnetic fields are encoded in F:

$$E_1 = F_{01}, \quad E_2 = F_{02}, \quad E_3 = F_{03}, \quad B_1 = F_{23}, \quad E_2 = F_{31}, \quad B_3 = F_{12} \tag{4.16}$$

To interchange $\vec{E}$ with $\vec{B}$, we define the Levi-Civita tensor ϵ_{abcd}. This is totally antisymmetric in its indices, so it vanishes when any two of the indices are equal. We set $\epsilon_{0123} = 1$. For any other permutation of indices, we get a value 1 if the permutation is even (i.e. involves an even number of index interchanges) and a value -1 if the permutation is odd. We write

$$\tilde{F}_{ab} = \frac{1}{2!}\epsilon_{abcd}F^{cd} \tag{4.17}$$

The change $F \to \tilde{F}$ gives the map (4.11); for example

$$\tilde{E}_1 = \tilde{F}_{01} = F_{23} = B_1 \tag{4.18}$$

We can now define a dual gauge field $\tilde{A}_a$ through

$$\tilde{F}_{ab} = \partial_a \tilde{A}_b - \partial_b \tilde{A}_a \tag{4.19}$$

While an electrically charged particle couples to the gauge field A_a as (4.1), a magnetically charged particle couples to $\tilde{A}_a$

$$S_{int} = q_m \int \tilde{A}_\mu d\xi^\mu \tag{4.20}$$

In string theory we have branes extended in p spatial directions, and gauge fields with indices $A_{\mu_1, \dots \mu_{p+1}}$. How should we define electric-magnetic duality in this situation? We proceed as follows:

(i) Let the spacetime have dimension D.

(i) A p-brane has a $p+1$ dimensional world volume, and couples to a gauge field $A_{\mu_1 \dots \mu_{p+1}}$ with $p+1$ indices, as in (4.10).

(ii) The field strength of A is $F_{\mu_1 \dots \mu_{p+2}}$, with $p+2$ indices.

(iii) To define the dual field strength, we use the Levi-Civita tensor in D dimensions. This gives

$$\tilde{F}_{\mu_1 \dots \mu_{D-p-2}} = \epsilon_{\mu_1 \dots \mu_{D-p-2}\mu_{D-p-1} \dots \mu_D} F^{\mu_{D-p-2} \dots \mu_D} \tag{4.21}$$

so $\tilde{F}$ has $D-p-2$ indices.

(iv) We define a dual gauge potential $\tilde{A}$ such that

$$\tilde{F}_{\mu_1 \dots \mu_{D-p-2}} = \partial_{\mu_1} \tilde{A}_{\mu_2 \dots \mu_{D-p-2}} + \text{cyclicpermutations} \tag{4.22}$$

where the permutations are defined with signs as in (**??**). This dual gauge field $\tilde{A}$ has $D-p-3$ indices.

(v) A gauge field with $D-p-3$ indices couples to a brane with $D-p-4$ space directions. Thus the dual of a p brane in D spacetime dimensions is a $D-p-4$ brane.

For the above example of electromagnetism, we have a spacetime dimension $D=4$ and pointlike electric charges, so $p=0$. The magnetic dual charges will have $p'=D-p-4=0$, so the magnetic monopoles will be pointlike as well.

Now consider type IIA string theory, where $D=10$. This theory has a D0 brane, with $p=0$. Its dual under electric-magnetic duality will be a brane with $p'=D-p-4=10-4=6$. We do indeed have a brane with this dimension in the theory: the D6 brane. Proceeding in this way, we find the electric-magnetic duality pairs:

$$\begin{array}{lccc} IIA: & D0 & \leftrightarrow & D6 \\ & D2 & \leftrightarrow & D4 \end{array} \tag{4.23}$$

$$\begin{array}{lccc} IIB: & D1 & \leftrightarrow & D5 \\ & D3 & \leftrightarrow & D3 \end{array} \tag{4.24}$$

Note that the D3 brane of IIB theory is self-dual: it carries both electric and magnetic couplings with equal strength.

Both IIA and IIB theories have the elementary string, for which we have

$$IIA, IIB: \qquad NS1 \quad \leftrightarrow \quad NS5 \tag{4.25}$$

Thus we see that the set of elementary objects that we have found in the theory neatly falls into pairs of electric-magnetic duals.

Lecture notes 5

M theory

5.1 M theory

We have seen that string theory has a logical and compelling structure. Elementary qualitative requirements – like the preservation of T and S dualities at the quant level – force a unique the particle and gauge field content for the theory, so that at the end the theory has no free parameters. One may however feel that the theory is quite complicated, given the large number of possible branes. We will now see that this complicated looking structure actually arises from something much simpler, if we use the perspective of 11-d M theory.

The dilaton ϕ was a scalar in string theory which signalled the existence of an extra dimension x_{11}; if this extra dimension was compactified to a circle, then we obtained IIA string theory where the dilaton described the size of this extra circle. One is therefore led to ask what each object in string theory looks like in the 11-d M theory. Let us start with the elementary string of string theory. What does this look like in M-theory?

A clue comes from the difference in metrics that we used to study M-theory and string theory. For the M-theory the metric was called $g_{\mu\nu}$. For string theory, we write

$$g_{11,11} = e^{C} \tag{5.1}$$

and define the string metric

$$g^S_{\mu\nu} = e^{\frac{C}{2}} g_{\mu\nu} \tag{5.2}$$

In the string metric g^S, the length of a string was a fixed value l_s everywhere. But if we used the M-theory metric g, then this length would depend on the local value of C. The metric measures lengths squared, so a factor $Exp[-C/2]$ in the metric gives a factor $Exp[-C/4]$ when measuring lengths; thus the length of the string using the M-theory metric g will be

$$l'_s = l_s e^{-\frac{C}{4}} \tag{5.3}$$

We can rewrite this fact in terms of the string tension. The tension of the string in the string metric is defined as

$$T_{NS1} = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2} \tag{5.4}$$

Using the M-theory metric this will become

$$T'_{NS1} = \frac{1}{2\pi l'^2_s} = T_{NS1} e^{\frac{C}{2}} \tag{5.5}$$

From (5.1) we see that $Exp[C/2]$ is proportional to the length of the direction x^{11}. Thus (5.5) says that the tension of the string seen in 10-d is linearly proportional to the length of the extra circle of M-theory. This suggests that we think of the 1-dimensional string as arising from a 2-dimensional sheet, where one direction of this sheet has been wrapped around x_{11}. This 2-d sheet is called the M2 brane.

We can write down the tension T_{M2} of this M2 brane. In the string metric, the length of x_{11} is

$$L_{11} = 2\pi g l_s \tag{5.6}$$

Thus the tension of the string would be

$$T_{NS1} = L_{11} T_{M2} = 2\pi g l_s T_{M2} \tag{5.7}$$

Using (5.4) we find

$$T_{M2} = \frac{1}{(2\pi)^2 g l_s^3} \tag{5.8}$$

where this tension is given in the string metric. We can of course convert it to the 11-d metric g, noting that T_{M2} has units of $(length)^{-3}$:

$$T'_{M2} = e^{\frac{3}{4}C} T_{M2} = e^{\phi} T_{M2} = \frac{1}{(2\pi)^2 l_s^3} \tag{5.9}$$

This does not involve the factor $Exp[C/2]$ governing the size of the x_{11} direction. This is exactly as it should be, because we want the M2 brane to be a fundamental object in 11-d M theory with no reference to any compactification.

5.1.1 Developing the theory

We can now uncover the structure implied by M-theory, as follows:

(A) We have seen that M-theory has an elementary object: the M2 brane. If we wrap this M2 on the compact circle x_{11}, we get the string $NS1$ of IIA string theory. But we can also choose to *not* wrap the M2 in this fashion – we can let both of its spatial directions lie along the spatial directions $x^1, \dots x^9$ of string theory. This gives a 2-brane in string theory, which we recognize as the D2 brane; in particular, the tension (5.8) agrees with the tension (**??**) found for the D2 brane.

(B) The M2 brane has a 3-dimensional worldvolume, and so will couple to a gauge field C_{ABC} in the 11-d M theory. If we wrap one direction of the M2 along x_{11} to get the string, then one direction of this worldvolume is along x_{11}. The other two directions lie in the 10 directions of string theory, and give the gauge field $B_{\mu\nu}$ coupling to the fundamental string. If we do not wrap the M2 along the x_{11} direction, then all components of C_{ABC} lie along the 10 directions of string theory, and give the gauge field $A_{\mu_1\mu_2\mu_3}$ coupling to the D2 brane.

(C) Let us now ask that M theory possess electric-magnetic duality. The dual of a $p = 2$ brane in a spacetime with dimension $D = 11$ will be a p' brane with

$$p' = D - p - 4 = 5 \tag{5.10}$$

Thus M-theory should have an M5 brane. We have two choices:

(i) We can wrap the M5 on the circle x_{11}; this gives the D4 brane of IIA string theory.

(ii) We can *not* wrap the M5 along x_{11}; this gives the NS5 brane of IIA string theory.

The gauge field $\tilde{C}$ dual to C couples to the M5 brane, and gives the gauge field couplings to the D4 and NS5 in string theory.

We have already seen that the D0 brane of IIA arises from a graviton moving along x_{11} in M-theory. This completes the content of IIA string theory, except for one object: the D6 brane. The source of this brane in M-theory is a very interesting object called a KK-monopole, and will play an important role in our understanding of black holes.

5.1.2 The KK-monopole

The D0 brane arises from the pure gravity theory of 11-d; it does not involve the particle content introduced through the M2 and M5 branes. We will now see that its dual under electric-magnetic duality is also an object in pure gravity – it arises as a topological soliton. This soliton is obtained by twisting the extra circle x_{11} in a certain way. Since this circle is involved in the idea of Kaluza-Klein higher dimensional theory, the monopole is called the Kaluza-Klein monopole, or the KK-monopole for short.

We proceed in the following steps:

(A) The structure of the KK monopole will not involve 6 of the noncompact spatial directions of M-theory. Thus let the metric in these directions be trivial

$$ds^2 \ \rightarrow \ \sum_{i=1}^{d} dx_i dx_i \tag{5.11}$$

Whatever structure we make in the remaining directions will stretch uniformly along these 6 spatial directions; this will generate a D6 brane in IIA string theory.

(B) In the remaining 3 noncompact directions, introduce polar coordinates r, θ, ϕ. The monopole described below will be a localized object centered around $r = 0$. For the next steps we will focus only on the directions r, θ, ϕ and x_{11}.

(C) Consider any given value of r. The directions θ, ϕ form a sphere S^2. Imagine dividing this S^2 into a north and a south hemisphere.

At each point of this S^2 we have a copy of the circle x^{11}; let us parametrize this circle by a coordinate $0 \leq \xi < 2\pi$. For all the circles over the North hemisphere, mark a point corresponding to $\xi = 0$. Do the same for the south hemisphere.

(D) We now wish to glue the North and South hemispheres at the equator. In this glueing, the x_{11} circle over each point of the equator on the North side will glue to the corresponding circle at the equator on the south side. The simplest possibility is to perform this gluing by aligning the $\chi = 0$ points on the North circles with the $\chi = 0$ points on the corresponding South circles. If we do this, we will recover the original flat spacetime of 10+1 dimensional M theory compactified on a circle. But we can make different gluing, as follows.

Start with the point $\phi = 0$ on the equator, and glue the North and South circles while aligning the points $\chi = 0$. Now move to a point $\phi > 0$. At this point, perform the gluing after a rotation of the South circle; thus the point $\chi = 0$ on the North will be aligned with the point $\chi = \phi$ on the South. Proceeding in this way all around the equator, we find that the point $\phi = 2\pi$ is glued with a 2π rotation between the North and South circles. This is of course equivalent to no rotation at all; thus it coincides with the gluing we started with at $\phi = 0$, as it should.

(E) The above gluing process twists the circle x_{11} nontrivially over the S^2 at radius r. We call this a Hopf fibration of a circle S^1 over S^2. If we had taken the trivial gluing mentioned above, then we would get a simple product space $S^1 \times S^2$. With the twist, the manifold we get from the directions θ, ϕ, x_{11} is topologically a 3-sphere S^3. This fact is not immediately obvious, but can be seen from the explicit metrics we will give below.

(F) While we have created a smooth change of topology at each value of $r > 0$, it may seem that we will get a singularity at $r = 0$. This is because the equator in the above construction will get smaller as $r \to 0$. The x^{11} circle will to rotate by 2π over this small length, so it will generate diverging gradients at $r = 0$.

But as it turns out, there is no singularity at $r = 0$. At each r the surface formed by θ, ϕ, x_{11} is topologically S^3. As we go to smaller r, the θ, ϕ directions become smaller. Suppose we also make the x_{11} circle shrink in length as $r \to 0$. Then we get a set of nested spheres S^3, with size that is getting smaller as $r \to 0$. This gives a smooth *four* dimensional space, where r is still the radial direction and the S^3 gives the angular sphere. Thus the full spacetime will be a smooth manifold with a topological 'knot' in it; this geometry is called the KK-monopole.

(H) We can make an anti-monopole by reversing the direction of the twist. Thus in step (D), the point $\chi = 0$ on the North will be aligned with the point $\chi = -\phi$ on the South.

5.1.3 The metrics of D0 and D6 branes

When we perform a dimensional reduction to 10-d string theory, the KK monopole will generate D6 brane. To see how this is the magnetic dual of the D0 brane, let us look at the metrics describing the D0 and D6 branes.

Metric of the D0 brane

The D0 brane arises from a massless graviton travelling at the speed of light in the direction x^{11}. There are two kinds of metrics that we can write

(i) The metric of a massless point particle moving along x^{11}.

(ii) The metric if a line of massless particles distributed uniformly along x^{11}, each moving along x^{11}.

The metric is known in each case; it is called the Aichelberg-Sexl metric. Taking the line of particles as in (ii) is called 'smearing the source' along the x^{11} direction. When a large number of particles in involved, as in black holes, (ii) is a more natural metric to consider. It is given by

$$ds^2 = -dt^2 + dx_{11}^2 + \frac{P}{r^7}(dt + dx_{11})^2 + \sum_{i=1}^{9} dx^i dx^i \tag{5.12}$$

where $r = (\sum_{i=1}^{9}(x^i x^i))^{\frac{1}{2}}$. The parameter P characterizes the strength of the source; the letter P stands for the fact that this source carries momentum in the x_{11} direction.

In the dimensionally reduced IIA theory, we have a gauge field A_μ arising from the metric components $g_{11\mu}$. At large values of r, we have

$$A_0 \approx g_{11,0} \approx \frac{P}{r^7} \tag{5.13}$$

Thus the gauge field A_μ arising from $g_{11\mu}$ is the field coupling to the D0 brane. The field strength corresponding to (5.13) will have the components F_{0r}, so the D0 is an electrically charged particle producing an electric field $\vec{E} = E(r)\hat{r}$.

The metric of the D6 brane

The field strength of the D0 brane had the components F_{0r}. The dual field strength $\tilde{F}$ defined as (4.21) will then have components only in the angular space directions, and the gauge potential $\tilde{A}$ can also be taken with components only in these angular space directions. Thus the magnetic dual of the D0 brane should produce a potential of this nature.

Let us now write down the metric which was described qualitatively in section 5.1.2:

$$ds^2 = \frac{1}{1+\frac{Q}{r}}[dx_{11}^2 + Q(1-\cos\theta)d\phi^2] + (1+\frac{Q}{r})[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \sum_{i=1}^{6} dx^i dx^i \tag{5.14}$$

This metric is a solution of the Einstein equations in vacuum; i.e., with no source of energy momentum. Let us note the relevant features of this metric:

(i) At large r, the gauge field has the component

$$A_\phi \approx g_{11,\phi} \approx Q(1-\cos\theta) \tag{5.15}$$

Thus its components are in the angular space directions, and in fact this is exactly the gauge field expected for a magnetic monopole.

(ii) As $r \to 0$, the length of the x^{11} direction vanishes:

$$\frac{1}{1+\frac{Q}{r}}dx_{11}^2 \approx \frac{r}{Q}dx_{11}^2 \tag{5.16}$$

so the x_{11} circle pinches off at $r=0$. But the direction x_{11} is not orthogonal to the ϕ direction, and we need to use alternative coordinates to make the physics clearer near $r=0$.

(iii) These alternative coordinates are:

$$\begin{array}{rcll} \tilde{r} & = & \sqrt{r}, & 0 < \tilde{r} \\ \tilde{\theta} & = & \frac{\theta}{2}, & 0 \le \tilde{\theta} < \frac{\pi}{2} \\ \tilde{y} & = & \frac{x^{11}}{2Q}, & 0 \le \tilde{y} < \frac{L}{2Q} \\ \tilde{\phi} & = & \phi - \frac{x^{11}}{2Q}, & 0 \le \tilde{\phi} < 2\pi \end{array} \tag{5.17}$$

In these coordinates the metric near $r=0$ becomes

$$ds^2 \approx 4Q[d\tilde{r}^2 + r^2(d\tilde{\theta}^2 + \cos^2\tilde{\theta}d\tilde{y}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2)] \tag{5.18}$$

which is the metric of R^4 in polar coordinates. Thus the solution (5.14) is regular at $r=0$.

5.1.4 The action of M-theory

We have seen that M-theory has a very simple structure. The only branes are M2 and M5. They both couple to one gauge field C_{ABC}; one electrically and one magnetically. The action thus involves just the metric and C_{ABC}

$$S = \frac{1}{(2\pi)^8 l_p^9}[\int d^{11}x\sqrt{-g}[R - \frac{1}{48}F_{ABCD}F^{ABCD}] + \frac{1}{6}\int d^{11}x \, \epsilon_{A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11}}C^{A_1A_2A_3}F^{A_4A_5A_6A_7}F^{A_8A_9A_{10}A_{11}} \tag{5.19}$$

Here F is the field strength of C. The last term is a Chern-Simmons (CS) term; its existence and precise coefficient can be deduced from T-duality applied to metrics with off diagonal components g_{ij}. Such CS terms will play an interesting role in the breaking of the no-hair theorem in string theory.

Despite the conceptual simplicity of M-theory, it is hard to compute with at the quantum level. This is because there is no small parameter that can be used to define a perturbation series. If we compactify a circle x^{11}, then the size of this circle gives the dimensionless coupling g, and we can do useful computations in IIA string theory in the domain $g \ll 1$.

Lecture notes 6

Bound states in string theory

We have seen that string theory requires a definite set of elementary objects, with definite tensions. To make black holes, we will have to make a bound state of a large number of these elementary objects. As we will now see, the rules for binding are also simple and intuitive, and the binding energies are given by simple expressions.

6.1 Bound state of D0 branes

We get a D0 brane of IIA string theory by taking a graviton h_{ij} and giving it momentum along x^{11}; thus the graviton has a wavefunction (eq.(**??**))

$$|\psi\rangle_1 \sim e^{i\frac{x^{11}}{R_{11}}} \tag{6.1}$$

where R_{11} is the coordinate radius of x^{11}. This graviton has energy and momentum

$$E_1 = P_1 = \frac{1}{R_{11}} \tag{6.2}$$

Now consider the state

$$|\psi\rangle_n \sim e^{in\frac{x^{11}}{R_{11}}} \tag{6.3}$$

This has momentum and energy

$$E_n = P_n = \frac{n}{R_{11}} \tag{6.4}$$

In IIA theory, this wavefunction has the charge of n D0 branes. Thus we find the following:

(i) There exits a bound state where n D0 branes bind together, for all values of n.

(ii) This bound state has zero binding energy, since

$$E_n - nE_1 = 0 \tag{6.5}$$

Such bound states are called 'threshold bound states', since they can be broken with zero expense in energy.

(iii) The graviton h_{ij} carried indices i, j, so the D0 brane had a variety of possible spin states. We could also replace the graviton by any of the other massless quanta, like B_{ij}, ϕ or the gauge fields $A_{i_1,\dots i_k}$. Overall one finds 128 such bosonic states and 128 fermionic states. Thus the D0 brane comes in a multiplet of 256 states.

This number of possibilities for spin does not depend on n; the wavefunctions (6.3) are made with the same quanta as the wavefunctions (6.1). Thus we find that the degeneracy of bound states with charge n remains 256 for all values of n.

Since all elementary objects can be mapped to each other by dualities, the degeneracy of the bound state of n branes of the same type will be

$$\mathcal{N} = 256 \tag{6.6}$$

6.2 Bound states of strings

We have seen that all elementary objects in string theory are related by T,S dualities. Thus the existence of threshold bound states of D0 branes implies that we get similar threshold bound states for *each* brane. Let us look at this bound state for the elementary string NS1, since in this case the bound state has a pictorial description which will be very useful later.

Take IIA string theory, and compactify the direction X^9 to a circle of length L. Let us wrap an elementary strings on this circle. The energy of this state will be

$$E_1 = T_{NS1}L \tag{6.7}$$

Now take the string and wrap it n times around the circle, before joining its endpoint to its starting point. This is called a 'multiwound' string. The energy of this string is

$$E_n = T_{NS1}nL = nE_1 \tag{6.8}$$

so we again get a threshold bound state.

6.3 The D2D0 bound state

Consider IIB string theory, and compactify two directions: x^8 to a circle of length L_8 and x^9 to a circle of length L_9. The compact directions thus form a torus T^2.

Wrap a D1 brane along x^8, and a D1 brane along x^9, as shown is fig.**??**(a). The total energy of this configuration is

$$E_T = T_{D1}L_8 + T_{D1}L_9 \tag{6.9}$$

In fig.**??**(b) we depict the bound state of these two D1 branes. It carries the same charges: winding of one unit along x^8 and winding of one unit along x^9.

Clearly, the D1 brane wound on the diagonal cycle of the T^2 is the lowest energy state with these charges; it has energy

$$E_{bound} = T_{D1}\sqrt{L_8^2 + L_9^2} \tag{6.10}$$

The binding energy is nonzero in this case

$$E_{binding} = E_T - E_{bound} > 0 \tag{6.11}$$

Each of the two initial D1 branes had 256 possible spin states (by eq.6.6)). The bound state is another D1 brane wrapped on a circle, so this also has the degeneracy

$$\mathcal{N} = 256 \tag{6.12}$$

We can again take this example and apply T,S dualities to obtain other bound states. For example, let us apply a T-duality in the direction x^8. This will bring us to IIA theory, with the following objects in then initial state

(i) The D1 brane along x^9 becomes a D2 brane wrapping x^8x^9. Let $L'_8, L'9$ be the lengths of the cycles after the duality. Then the energy of the D2 is

$$E_{D2} = T_{D2}L'_8L'_9 \tag{6.13}$$

(8) The D1 brane along x^8 becomes a D0 brane. Its energy is

$$E_{D0} = T_{D0} \tag{6.14}$$

(iii) The energy of the bound state is

$$E_{bound} = \sqrt{E_{D2}^2 + E_{D0}^2} \tag{6.15}$$

This bound state again has a degeneracy

$$\mathcal{N} = 256 \tag{6.16}$$

If we had more compact directions, then we could do additional T-dualities and get other brane bound states from the D2D0 bound state. A T-duality along x^7 will give D3D1, and similarly we can get D4D2, D5D3, and D6D4. All these combinations will have branes whose dimensions differ by 2: We have a p brane bound to a $p+2$ brane, with all the directions of the p brane contained in the $p+2$ brane. All such bound states will have their energy given in the form (**??**) and the degeneracy (6.16).

6.4 The D4D0 bound state

Take IIA string theory for concreteness, and wrap compactify x^9 to a circle of length L_9. Wrap an elemnetary string NS1 along this circle; this has energy

$$E_{NS1} = T_{NS1} L_9 \tag{6.17}$$

Now take a momentum mode P along x^9; this has energy

$$E_P = \frac{2\pi}{L_9} \tag{6.18}$$

We wish to find the bound state of these two objects. This bound state will represent momentum moving along a string. There is a simple geometrical picture of this bound state: the string carries the momentum in the form of travelling waves, as depicted in fig.**??**(b). Recall that the string has no longitudinal vibration modes; thus the travelling wave involves only transverse deformations of the string.

We had seen that small transverse deformations X^i were described by a massless scalar field defined on the string. We can take an excitation of this scalar field with waveform

$$|\psi\rangle \sim e^{2\pi i \frac{x^9}{L_9}} \tag{6.19}$$

This has energy and momentum along x^9 given by

$$E = P = \frac{2\pi}{L_9} \tag{6.20}$$

The total energy of the bound state is then

$$E_{bound} = T_{NS1} L_9 + \frac{2\pi}{L_9} \tag{6.21}$$

We did this computation for small amplitiudes of the transverse deformation, but it is actually an exact expression for all amplitudes of deformation. We now observe that

$$E_{bound} = E_{NS1} + E_P \tag{6.22}$$

so we again have a bound state at threshold.

We have seen that this NS1P bound state can be dualized to D4D0, or D5D1. Thus a D1 brane will bind at threshold to a D5 brane is it lies in the plane of the D5 brane.

What is very interesting is the degeneracy of such bound states, which we now investigate. Suppose we have one NS1 wrapped as in (6.17), but we bind to it n units of the momentum P. Thus the value of the energy and momentum along x^9 is

$$E = P = \frac{2\pi n}{L_9} \tag{6.23}$$

The wavefunction (6.19) carried one unit of P, but we can also have wavefunctions that carry any given integer k of momentum:

$$|\psi_k\rangle \sim e^{2\pi i k \frac{x^9}{L_9}}, \quad k = 1, 2, \ldots \tag{6.24}$$

Such an excitation has

$$E = P = \frac{2\pi k}{L_9} \tag{6.25}$$

The state (6.24) has a simple interpretation. The excitations of the D1 are described by transverse vibrations. Consider vibrations in the harmonic k; i.e., a vibration with wavelength $\lambda = L_9/k$. We can quantize these vibrations, and then (**??**) described a single quantum of excitation with this wavenumber.

We now see that we can obtain the E and P in (6.23) by taking an excitation (6.24) with $k = n$. But we can also get the same overall E, P by taking any set of excitations $k_1, k_2, \ldots k_j$ where

$$k_1 + k_2 + \ldots k_j = n \tag{6.26}$$

Each such partition of n gives a possible state with the quantum numbers (6.23). Thus the bound state of NS1 and P has a degeneracy that can be very large if n is large. The number of partitions $\mathcal{N}$ for $n \gg 1$ is given by

$$\mathcal{N} \sim e^{2\pi\sqrt{\frac{n}{6}}} \tag{6.27}$$

Thus the degeneracy of the NS1P bound state can be very large for large values of n. This fact will be crucial for us, since this large degeneracy will furnish the entropy of black holes. So let us compute the degeneracy more carefully. We have not one but 8 transverse directions of oscillation. Each direction gives a different 'flavor' of excitations, so the total momentum n will have to be divided among these flavors. We have 8 fermionic excitations as well, and it turns out that each fermion acts like half a boson. Thus the total effective number of flavors is

$$f = f_B + \frac{1}{2} f_F = 8 + 4 = 12 \tag{6.28}$$

The total number of possible states are obtained by multiplying the number of states for each flavor. The excitation level for each flavor is n/f. This gives for the number of states

$$\mathcal{N} \sim \left(e^{2\pi\sqrt{\frac{n}{6f}}}\right)^f = e^{2\pi\sqrt{\frac{fn}{6}}} \tag{6.29}$$

Setting $f = 12$ gives

$$\mathcal{N}_{NS1P} \sim e^{2\pi\sqrt{2}\sqrt{n}} \tag{6.30}$$

We have taken n units of P charge, but thus far kept only one unit of NS1 charge. We will now see that when we have n_1 units of NS1 winding, we get a remarkable feature called fractionation, which will be vital in our study of black holes.

6.5 Fractionation

Let us make a bound state with n_1 units of NS1 winding along x^9, and n_p units of momentum along x^9. Let the length of the x^9 We can make an intuitive picture of this bound state, as follows:

(i) The n_1 units of NS1 winding join up into a multiwound string, with total length

$$L_T = n_1 L \tag{6.31}$$

(ii) The momentum P is again carried by transverse vibrations of this string. The wavefunctions (6.24) were periodic under $x^9 \rightarrow x^9 + L$. But now the waveform need to be periodic only under

$$x^9 \ \rightarrow \ x^9 + L_T \tag{6.32}$$

The wavefunctions for the excitations then have the form

$$|\psi_k\rangle \sim e^{2\pi i k \frac{x^9}{L_T}} \tag{6.33}$$

Such an excitation has

$$E = P = \frac{2\pi k}{L_T} = \frac{2\pi k}{n_1 L} \tag{6.34}$$

(iii) This is a very interesting phenomenon, so let us examine it in more detail. A momentum mode P by itself comes in units of

$$E = P = \frac{2\pi}{L} \tag{6.35}$$

But when it is bound to n_1 strings, it can exist in *fractional* units, which are multiples of

$$E = P = \frac{2\pi}{n_1 L} \tag{6.36}$$

Thus a P mode 'breaks up' into n_1 pieces when it is bound to n_1 strings. These fractional pieces can then join up in any way: if k of these fractional units join up, then we get an excitation with the quantum numbers (**??**).

(iii) We have n_p units of momentum, which give

$$P = \frac{2\pi n_p}{L} = \frac{2\pi n_1 n_p}{L_T} \tag{6.37}$$

Note that n_p must be an integer; we cannot have a *net* fractional number of units of momentum. This is because the integrality of n_p follows from a basic quantum mechanical principle: the full wavefunction of the state should be invariant under

$$x^9 \rightarrow x^9 + L \tag{6.38}$$

Thus while individual excitations of the string can have fractional units of p charge, the physically allowed states are only those where the net momentum from all excitations is an integer.

(iv) We can now count the degeneracy of the bound state of n_1 strings and n_p units of momentum. The states with these quantum numbers have excitations of the form (6.34), with

$$\sum_j k_j = n_1 n_p \tag{6.39}$$

Thus the degeneracy is (6.30)

$$\mathcal{N} \sim e^{2\pi\sqrt{2}\sqrt{n_1 n_p}} \tag{6.40}$$

This degeneracy grows very rapidly with the charges n_1, n_p, and will form the prototypical example of black hole entropy.

6.6 Nonextremal bound states

The bound states we studied above had $E = P$. Since P is the momentum charge,, we can say that we have $E = Q$, which gives an extremal or BPS state. But we can also make bound states that are not extremal. The wavefunctions (6.24) has momentum in the positive direction x^9, but we can consider excitations with wavefunctions

$$|\psi_k\rangle \sim e^{-2\pi i k \frac{x^9}{L_9}}, \quad k = 1, 2, \dots \tag{6.41}$$

These describe travelling waves moving in the negative x^9 direction, and have

$$E = -P = \frac{2\pi k}{L_9} \tag{6.42}$$

We can take some excitations moving in the positive x^9 direction and some moving in the negative x^9 direction. Then we get states with

$$E > |P| \tag{6.43}$$

which corresponds to nonextremal states. Recall that even though the individual excitations can have fractional momenta, the overall momenta must be an integer.

We can now compute the 'energy gap' of the NS1 bound state: the minimum energy ΔE required to excite the bound state without addition of any net charge. First consider the singly wound string; i.e., $n_1 = 1$. Since we want no net P charge, we add the lowest allowed energy excitations in each of the directions along the string, getting

$$\Delta E_1 = \frac{2\pi}{L} + \frac{2\pi}{L} = \frac{4\pi}{L} \tag{6.44}$$

Now consider the NS1 bound state with $n_1 > 1$. We find

$$\Delta E_{n_1} = \frac{2\pi}{n_1 L} + \frac{2\pi}{n_1 L} = \frac{4\pi}{n_1 L} \tag{6.45}$$

Thus we see that the energy gap drops when n_1 in increased; i.e., it needs less energy to excite a bound state of many strings compared to a single string. This fact will be vital to understanding how large bound states – representing large black holes – can absorb particles of very long wavelengths.

6.7 Summary

We see that in many ways string theory is simpler than the field theories of other interactions. We still do not have a good way of computing the binding energy of quarks and gluons to obtain the mass of a proton. But the binding energy of string states is given by simple geometric expressions, and a few special examples can be used to study all other cases through the use of T,S duality maps.

Lecture notes 7

The open string picture of D-branes

We have found the D-branes of string theory by enforcing the T and S dualities as exact symmetries. But historically, they were discovered in a different way: by looking at the theory of *open* strings.. This perspective adds a lot of insight into the physics of D-branes, as we will now see.

7.1 Open strings

So far we have thought of strings as closed loops; vibrations can then run clockwise or anticlockwise around this loop. But we can also consider an open string, which is just a line segment of finite length. What does such a string represent?

Before we can answer this question, we note a potential problem with the open string. Just like with the closed string, the excitations of the open string will be waves that travel along the string. But what happens when a traveling wave reaches the end of the string? If its energy just flows off the string, then the open string would not be a consistent object by itself; we would have to consider the open string along with whatever entity took the energy that flowed off the string.

There are two ways to prevent energy flowing off the end of the string:

(i) We can let the endpoint move at the speed of light. This is called the Newmann (N) boundary condition. As the travelling wave approaches the end of the string, its motion slows down due to time dilation from the high velocity of the string near its endpoint. Thus the energy never reaches the endpoint itself.

(ii) We can require that the endpoint be fixed at a point in space. This is called the Dirichlet (D) boundary condition. Since the endpoint does not move, the travelling wave just reflects off the endpoint and returns to the interior of the string.

An open strings with N boundary conditions would be an elementary object moving in spacetime, just like any other object. But what would be the meaning of a string with D boundary conditions? This string has its endpoints fixed at a given location, say $\vec{x} = 0$. Choosing such a location would break the translation symmetry of spacetime, which seems an unnatural thing to do. Yet the open string with D boundary conditions exists, and if we find no reason to exclude it, then we must incorporate its states into our theory.

In 1994 Polchinski discovered an elegant interpretation for strings with D boundary conditions. Suppose there was a heavy object sitting at the location $\vec{x} = 0$. Then the open string with ends at $\vec{x} = 0$ would represent an excitation of this heavy object. Since the excitations are given by open strings with Dirichlet boundary conditions, we call this object a Dirichlet-brane, or D-brane for short.

In the above instance the object would be a D0 brane; since the open strings all end at a point $\vec{x} = 0$, the heavy object must be a pointlike object at $\vec{x} = 0$. But we can also obtain other branes as follows. We require that the endpoints of the open string have $x^1, \dots x^8 = 0$, but that x^9 be arbitrary. Thus the end points can slide along a 1-dimensional line, giving a D1 brane along x^9. We say that we have D boundary conditions along $x^1 \dots x^8$, and N boundary conditions along x^9.

By choosing a different number of directions to be of D type, we get branes of different dimensions. A careful examination of the properties of these D-branes reveals that ew get D0,D2,D4,D6 brane sin IIA string theory and D1,D3,D5 branes in IIB theory. Their tensions are also given by the values (**??**) that we had found by using T,S dualities.

The discovery of D-branes was welcomed because it filled a hole in the overall picture of string theory. The states of the elementary string gave the gauge fields B_{ij}, as well as the gauge fields $A_{i_1,\dots i_{p+1}}$ for various p. The gauge field B_{ij} coupled to the elementary string NS1. But there seemed to be no object that coupled to the $A_{i_1,\dots i_{p+1}}$.

It is not a priori required that a gauge field in a theory have charged particles to couple to it. For example we can imagine Maxwell"s theory of $\vec{E}, \vec{B}$ fields in a world with no charged particles like electrons. But it made an elegant picture when the D-branes were found with just the right dimensions to couple with the gauge fields of the theory.

The D-branes have tensions $T \sim 1/g$. Thus when we do perturbation theory around $g = 0$ they appear as very heavy objects. This is why they were not found earlier, and were eventually discovered through their possible excitations. These excitations have low energy, and so are captured by open string states.

Let us now study the open string states in more detail.

7.2 Open strings and gauge theory

Consider IIB theory and take a D3 brane lying along the space directions x^1, x^2, x^3. The excitations of this brane will be given by open strings whose endpoints have N boundary conditions along x^1, x^2, x^3 and D boundary conditions for $x^4, \dots x^9$.

Just like the closed string, the open string has an infinite tower of energy eigenstates. The states of most interest however are the states with lowest energy. In many ways the open string behaves like half a closed string. Instead of the left and right moving waves on the closed string, we have just one set of vibration modes for the open string; the modes reflect off the ends of the string and are therefore described by standing waves. There is again a Casimir energy

$$E = -\frac{1}{2}\frac{1}{l_s} \tag{7.1}$$

The lowest allowed states are given by acting with a fermion vibration mode ψ^i which adds energy $E = \frac{1}{2l_s}$, which makes the total mass

$$m^2 = 0 \tag{7.2}$$

This massless excitation moves at the speed of light along the D3 brane; let this motion be along the direction x^1. The fermion index i takes 8 transverse values, $i = 2, \ldots 9$. These massless open strings attached to the brane have the following interpretation:

(i) The open strings with $i = 4, \ldots 9$ represent a transverse deformation of the brane in the direction x^i. This transverse deformation travels along the brane at the speed of light.

(ii) The open strings with $i = 2, 3$ represent gauge fields A_i along the D3 brane. We recall that gauge field excitations are massless and transverse, so A_i should have an index along $i = 2, 3$ but not along $i = 1$.

From our understanding of the action of D-branes we expected transverse vibrations to move along the brane at the speed of light. What we now learn in addition is that the D-branes carry a gauge field along their surface.

In terms of this gauge field, we can obtain an intuitive picture of some brane bound states. Consider the D2 brane, wrapped along the directions x^1, x^2. Let these directions be compactified to circles with lengths L_1, L_2 respectively. There is a gauge field A on the surface of theD2 brane. If we set $A = 0$, the we get just the unexcited D2 brane. But we can also have a nonzero field strength F_{12} for this gauge field. There is a natural quantization condition for F:

$$\int dx^1 dx^2 F = 2\pi n \tag{7.3}$$

where n is an integer. Such an F on the D2 describes n D0 branes bound to the D2 brane, and analysis of the D-brane action gives the energy (6.15) for this bound state.

7.3 Multiple D-branes

So far we have considered the excitations of a single D-brane. The most remarkable features of D-branes arise, however, when we consider more than one D-brane.

For concreteness take IIB theory, and a set of n D3 branes. We let the indices $a, b \dots$ run over $1, \dots n$. Let each brane stretch along x^1, x^2, x^3, and have the position $\vec{x} = 0$ in the transverse space $x^4, \dots x^9$.

The open string has two end points. Let the first endpoint be on brane a and the second endpoint on brane b. Thus the open string is labelled by an index pair (ab). This is true for all states of the open string. But let us for the moment focus on the massless states discussed above. We have the following:

(i) The transverse excitations x^i are now given by a matrix x^i_{ab} with indices $a, b = 1, \dots n$.

(ii) The gauge field excitations A_i with indices along the brane are now given by matrices A^i_{ab}.

A matrix valued gauge field has a natural interpretation: it is the gauge field for a non-abelian gauge theory. In fact we find that the entire low energy dynamics of the set of n D3 branes is given by such a gauge theory. The restriction to low energies implies that we only excite the massless open strings. The open string states form a supersymmetric set, so we have as many fermionic states as bosonic states. The supersymmetric theory is called $N = 4$ Yang-Mills theory, where the term $N = 4$ specifies the amount of supersymmetry in the theory. In such a theory, we have the gauge fields A^i_{ab}, the fermionic fields, and a set of scalars that exactly match onto the x^i_{ab}. The action governing the low energy dynamics of n D3 branes is then just the $N = 4$ super-Yang-Mills action

$$S = \tag{7.4}$$

It is remarkable that the entire low energy dynamics of the branes is captured in such an elegant form. But even more striking is the physical implication of such an action. In ordinary particle theory, each particle has 3 degrees of freedom from its possible translations, and so n particles will have $3n$ degrees of freedom. With D-branes, the diagonal elements x^i_{aa} describe the motion of the ath brane in the transverse direction i; this is analogous to the motions permitted for ordinary particles. But we also have the off diagonal elements in x^i_{ab}, so that the total number of degrees of freedom grows as $\sim n^2$. Thus the entire concept of position x^i has been generalized to a matrix. Novel features like this will make brane dynamics very different from particle dynamics when n is large.

7.4 S-duality

We have seen that if we have a D0 brane, then a T-duality converts this to a D1 brane. But a D1 brane is a 1-dimensional object, just like the string of string theory. Thus we have *two* 1-dimensional extended objects, and we can ask: is there any relation between these two objects? In particular, is there a symmetry which allows us to interchange thesde two objects, so that either one can be considered as the 'fundamental string' of string theory? We will see that the answer is yes; the relevant symmetry is called S-duality. Let us now study this duality, which along with T-duality, generates the full set of dualities of string theory.

The elementary string had a tension

$$T_{NS1} = \frac{1}{2\pi l_s^2} \tag{7.5}$$

while the D-sting has tension

$$T_{D1} = \frac{1}{2\pi l_s^2 g} \tag{7.6}$$

These do not seem to be related by a symmetry, but as we will now see, that is just a reflection of the fact that we are using the string metric $g^S_{\mu\nu}$ to compute the tensions. The string metric arose as the natural one to exhibit T-duality which mapped a graviton to a fundamental string (NS1); thus the string metric does not treat the NS1 and D1 strings symmetrically. We will see that when we use the Einstein metric, the two tensions do have an immediately recognizable symmetry. But before we do that, let us pause for a moment to clarify the meaning of measuring the tension in different metrics.

7.4.1 Measuring tensions

Consider a particle with mass m. We think of m as being a certain number when measured with a certain choice of units. But how do we set up these units from first principles, especially in a theory of gravity where spacetime is curved and no standard choice of coordinates is available?

There are three steps in defining units:

(a) First we put coordinates on our manifold. These are just numbers, used to label different points of spacetime. thus if we have a compact direction x, we map choose coordinates so that its coordinate range is

$$0 \le x < A \tag{7.7}$$

where A is just a number, with no physical meaning by itself.

(b) We now take a metric g_{ab} on this spacetime, which defines lengths through $ds^2 = g_{ab}dx^a dx^b$. The length of the coordinate interval (7.7) becomes

$$L = (g_{xx})^{\frac{1}{2}} A \tag{7.8}$$

This is a physical length. We can change the coordinate length A, which will induce a compensatory change in the metric g_{xx}, but the value of L will remain unchanged in this process. Note however that so far the number L has no relation to any physical object in the theory.

If we have a scalar field ϕ in our theory, then we can define different metrics

$$g^{\alpha}_{ab} = e^{\alpha\phi} g_{ab} \tag{7.9}$$

which will give different values L_α in place of (7.8).

(c) Now consider a physical object in the theory, for example a string. In its ground state, the vacuum fluctuations of this string give it a certain nonzero size. Define some way to measure this size, and call it l_s, the string length.

We can now measure lengths in units of this string length l_s. For example if we have a mass m, then its Compton wavelength Δx through

$$m\Delta x = 2\pi \tag{7.10}$$

If $\Delta x = l_s$, then we can write

$$m = \frac{2\pi}{l_s} \tag{7.11}$$

getting a mass that is 'string scale'.

(d) Let us now relate step (b) to step (c). Consider a string in its ground state at some point x_1. Let its coordinate length be Δx. Using the metric g_{ab}, this corresponds to some length L_1. But now consider a string in its ground state at a different point x_2. Again measure the length of this string; let it be L_2. The question now is: will the numbers L_1 and L_2 be the same?

The answer is: in general they will *not*. Suppose the tension T depended on ϕ, and ϕ had different values at x_1, x_2. Then the size of the string would be different at these two points, and the string length l_s defined through T would be different as well.

But we can choose a value $\alpha = \alpha_S$, and use the corresponding metric

$$g^{S}_{ab} = e^{\alpha_S\phi} g_{ab} \tag{7.12}$$

to measure lengths. Suppose that with this choice, the length L^S of the string in its ground state was the *same* at all points x. Then this metric would be particularly suited to studying strings, and would be called the 'string metric'.

(e) We could consider the D-string instead, and use its size to define a 'D-string length' l_d through $T_{D1} = (2\pi l_d^2)^{-1}$. We find

$$l_d = l_s g^{\frac{1}{2}} \tag{7.13}$$

Since g can be different at different points, the ratio l_d/l_s is not in general a constant. We can define a 'D-string metric' where l_d has the same value everywhere, but this metric will be different from the string metric.

(f) Suppose we have found the mass of a particle in units of the string length

$$m = \frac{a}{l_s} \tag{7.14}$$

where a is a pure number. Now suppose we define a metric different from the string metric

$$g^{\beta}_{ab} = e^{\beta\phi} g_{ab} \tag{7.15}$$

We have see that the length of a string L^{β} measured using this metric will be different at different points in this new metric

$$L^{\beta} = e^{\frac{\beta}{2}\phi} l_s \tag{7.16}$$

But we can define a new fundamental length, whose length l is related to the string length l_s through

$$l = e^{-\frac{\beta}{2}\phi} l_s \tag{7.17}$$

Then the length of this object will appear to be the same everywhere when we use the metric g^{β}_{ab}. We can now express the mass m of eq.(7.14) in terms of this new length l

$$m = \frac{a}{l_s} = \frac{e^{-\frac{\beta}{2}\phi} a}{l} \equiv \frac{a'}{l} \tag{7.18}$$

Thus the mass measured in the new units l is related to the mass measured in the old units l_s by

$$\frac{a'}{a} = e^{-\frac{\beta}{2}\phi} \tag{7.19}$$

The tension of a p brane has units $(length)^{-(p+1)}$, so it changes as

$$\frac{T'}{T} = e^{-\frac{(p+1)}{2}\beta\phi} \tag{7.20}$$

Let us now apply this to the derivation of S-duality. The type IIB action is given in (**??**). We define the Einstein metric

$$g^{E}_{ab} = e^{-\frac{1}{2}\phi} g^{S}_{ab} \tag{7.21}$$

This gives the action (**??**)

$$S = \frac{B}{16\pi G} \int d^{10}x \sqrt{-g^E} \left([R]_E - \frac{1}{2}\phi_{,c}\phi^{,c} \right) \tag{7.22}$$

This action has the symmetry

$$\begin{aligned} \phi &\rightarrow -\phi \\ g^{E}_{ab} &\rightarrow g^{E}_{ab} \end{aligned} \tag{7.23}$$

This is a symmetry of then classical action. For it to be a symmetry of the full quantum theory, it must also be a symmetry of the objects in the theory; i.e. a symmetry of the brane tensions. The change (7.21) gives $\beta = -\frac{1}{2}$ in (7.15). The tension (7.5) of the NS1 brane (the elementary string) becomes, in the Einstein frame

$$T'_{NS1} = e^{\frac{1}{2}\phi} T_{NS1} = \frac{1}{(2\pi)} g^{\frac{1}{2}} \tag{7.24}$$

and the tension (7.6) of the D1 brane becomes

$$T'_{D1} = e^{\frac{1}{2}\phi} T_{D1} = \frac{1}{(2\pi) g^{\frac{1}{2}}} \tag{7.25}$$

Under the symmetry $\phi \to -\phi$, we get $g \to g^{-1}$, and we see that

$$T'_{NS1} \leftrightarrow T'_{D1} \tag{7.26}$$

Thus the fundamental string and the D-string get interchanged under S-duality.

Now consider the D3 brane of the IIB theory, which has the tension

$$T_{D3} = \frac{1}{(2\pi)^3 g} \tag{7.27}$$

In the Einstein frame, this tension becomes (using (7.20)

$$T'_{D3} = e^{\phi} \frac{1}{(2\pi)^3 g} = \frac{1}{(2\pi)^3} \tag{7.28}$$

Thus under the S-duality map $g \to g^{-1}$, the D3 brane remains unchanged.

Finally, consider the D5 brane, which has the tension

$$T_{D5} = \frac{1}{(2\pi)^5 g} \tag{7.29}$$

In the Einstein frame, this tension becomes (using (7.20)

$$T'_{D5} = e^{\frac{3}{2}\phi} \frac{1}{(2\pi)^3 g} = \frac{1}{(2\pi)^3} g^{\frac{1}{2}} \tag{7.30}$$

Under an S-duality $g \to g^{-1}$, this changes to

$$T' = \frac{1}{(2\pi)^3} g^{-\frac{1}{2}} \tag{7.31}$$

Thus we need a new object in the theory which is a 5-brane, and which has a tension T' in the Einstein frame. We will call this object the NS 5-brane. Using (7.20), we can convert its tension (7.31) back to the string frame

$$T_{NS5} = T' e^{-\frac{3}{2}\phi} = \frac{1}{(2\pi)^3 g^2} \tag{7.32}$$

In quantum field theory, and object with a mass scaling with the coupling as g^{-2} is called a soliton. Such objects arise as localized smooth solutions of the classical field equations. Thus the NS5 brane is sometimes called the solitonic brane of string theory.

7.5 The D1D5 bound state

We start with type IIB string theory, which lives in 10-d. We compactify five directions to circles. We will treat one of these directions differently from the rest, so we term the compactification $t^4 \times S^1$. The T^4 is formed by the directions $x^7, \ldots x^9$, while the S^1 is along x^5. The directions $x^0 \equiv t$ and $x^1, \ldots x^4$ remain noncompact.

We take a bound state of two kinds of charges:

(i) We wrap an elementary string (i.e., and NS1 brane) along the S^1 direction, with winding number n_w.

(ii) We have n_p units of momentum along the S^1.

As we have seen above, this NS1-P bound state is described by a string carrying vibrations. We will now perform a set of duality maps:

(a) We perform an S-duality. The NS1 branes becomes D1 branes, while the momentum mode P remains unaffected. Thus we get a D1-P bound state in IIB theory.

(b) We perform T-dualities along the directions x^6, x^7, x^8, x^9. Under each such duality, the D1 brane acquires an additional direction, so that we end up with D5 branes wrapping along x^5, x^6, x^7, x^8, x^9. The P modes remain unaffected, so we get a D5-P bound state. Since we have performed an even number of T-dualities, we remain in IIB string theory.

(c) We perform an S-duality. This changes the D5 branes to NS5 branes, while the P mode remains unaffected. Thus we get an NS5-P bound state in IIB theory.

(d) We perform a T-duality in the direction x^5. The NS5 branes are unaffected by this duality, while the P modes change to NS1 along x^5. Thus we get an NS5-P bound state, where the NS5 extend along x^5, x^6, x^7, x^8, x^9, and the NS1 extend along x^5. The T-duality has moved us to IIA theory, so we get a NS5-NS1 bound state in the IIA theory.

Thus we see that we can change from one bound state – NS1-P – to another bound state – NS5-NS1 – by a sequence of dualities. We have already studied the NS1-P bound state in some detail. Let us use the above dualities to get a heuristic picture of the NS5-NS1 bound state:

(i) We started with n_w units of winding and n_p units of momentum. Thus in the NS5-NS1 bound state the number of 5-branes and 1-branes is given by

$$\begin{aligned} n_5 &= n_w \\ n_1 &= n_p \end{aligned} \tag{7.33}$$

(ii) In the NS1-P bound state, each unit of momentum broke up into n_w fractional units, with each fractional unit being $1/n_w$ of a full unit of momentum. Thus after the dualities, each NS1 brane should break up into n_5 fractional units, with each fractional unit being a string with $1/n_5$ times the tension of a full NS1.

(iii) In the NS1-P system the total number of fractional momentum units was $N = n_w n_p$. In Similarly, in the NS5-NS1 system the number of fractional NS1 branes will be

$$N = n_5 n_1 \tag{7.34}$$

(iv) In the NS1-P system the N fractional units of momentum were grouped into sets containing k_i fractional units

$$k_1 + k_2 + \cdots = N \tag{7.35}$$

Here the number k_i described the oscillator $a^{\dagger,\mu}_{k_i}$ acting on the unexcited state of the string.

In the NS5-NS1 system the fractional strings will be similarly grouped into sets containing k_i fractional strings, with

$$k_1 + k_2 + \cdots = N \tag{7.36}$$

Each group with k_i strings is a 'bound unit' of fractional strings. With ordinary strings, we had a simple picture of their bound state: we just obtained a multi-wound string. Thus the group with k_i fractional strings will be described by a fractional string that winds k_i times around the S^1 direction x^5 before closing. Each such bound unit will be called a *component string*, to distinguish it from an ordinary string which lives outside the 5-branes.

(v) In the NS1-P system, each oscillator $a^{\dagger,\mu}_{k_i}$ also carried a polarization $\mu = 1, \ldots 8$ which ranged over the 8 directions transverse to the NS1. In the NS5-NS1 system, each component string will carry a similar polarization. It will however be more convenient to use a different notation for this polarization, as described below.

There are 8 spatial directions transverse to the direction x^5. With out choice of compactification, these 8 directions are broken into two sets: the 4 directions $x^7 \ldots x^9$ along the T^4 and the 4 noncompact directions $x^1, \ldots x^4$. The 8 possible values for the vector index μ can be broken into a 4-component vector in the first set, and a 4-component vector in the second set.

Consider the 4-component vector in the T^4. This vector is a representation of the rotation group $SO(4)_I$, where the subscript I denotes the fact that these are 'internal' directions; i.e., directions along the compact T^4. The group $SO(4)_I$ can be decomposed as

$$SO(4)_I = SU(2)_1 \times SU(2)_2 \tag{7.37}$$

In this decomposition, the 4-component vector becomes a spin $\frac{1}{2}$ spinor under each of the two $SU(2)$ groups. Writing the corresponding spinor indices as $A, \dot{A}$, we label the 4 different allowed states as $|\psi^I_{A\dot{A}}\rangle$.

Now consider the 4-component vector in the noncompact directions. This vector is a representation of the rotation group $SO(4)_E$, where the subscript E denotes the fact that these are external (i.e., noncompact) directions. The group $SO(4)_E$ can be decomposed as

$$SO(4)_E = SU(2)_L \times SU(2)_R \qquad (7.38)$$

Here the subscripts L, R stand for 'left' and 'right' respectively; the reason for this nomenclature will be explained later. Again, the 4-component vector becomes a spin $\frac{1}{2}$ spinor under each of the two $SU(2)$ groups. Writing the corresponding spinor indices as $\alpha, \dot{\alpha}$, we label the 4 different allowed states as $|\psi^E_{\alpha\dot{\alpha}}\rangle$.

Thus we can use our picture of the NS1-P bound state to get a heuristic picture for the NS5-NS1 bound state. We depict these pictures in fig.**??**. In fig.**??**(a), we have the NS1-P system, where the multiwound NS1 is carrying the P charge as vibrations. Th vertical direction is x^5, while the horizontal direction describes $x^1, \ldots x^8$. The vibrations are described by an excitation of the string

$$|\psi\rangle = a^{\dagger,\mu_1}_{k_1} \ldots a^{\dagger,\mu_n}_{k_n}|0\rangle \qquad (7.39)$$

In Fig.**??**(b) we depict the corresponding $NS5 - NS1$ state. The vertical direction is again x^5, while the horizontal direction depicts the 4 directions $x^7, \ldots x^9$ along the T^4. The oscillator $a^{\dagger,\mu_1}_{k_1}$ in the NS1-P system gives a component string with winding k_1, which is shown as a multiwound strand in the figure. The polarization μ_1 of the oscillator can be encoded in the spinor components $A\dot{A}, \alpha\dot{\alpha}$, and this information is denoted by a spin arrow on the component string. The total winding of all components strings is $N = n_5 n_1$.

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