TOPIC V

BLACK HOLES IN STRING THEORY

Lecture notes 1

Making black holes

How should we make a black hole in string theory?

A black hole forms when a large amount of mass is collected together. In classical general relativity, the resulting gravitational field then deforms spacetime to create a horizon. To make a black hole in string theory, we must use the elementary excitations – gravitons, strings, branes etc. – present in the theory. We should put a large number of these excitations together, and see if we get a black hole.

The simplest excitation is the graviton. But the graviton is massless, and so moves at the speed of light. While a black hole can be made from a sufficient number of massless particles, it is easier to start with excitations that are massive, since these can be made to stay at a given location.

String theory has extended objects like strings and branes. In their lowest energy state, the force of tension makes such objects collapse to a point, and we again get massless excitations like gravitons and gauge fields. To get a massive excitation, we should somehow stretch these objects to a nonzero size. The simplest way to do this is to compactify some directions of space. We can then wrap the extended object on these compact directions, causing it to have a nontrivial extent, and thus a nonzero mass.

Let us now follow this path, and see if we get black hole.

1.1 Wrapped strings

We start as follows:

(i) String theory lives in 9+1 spacetime dimensions. We compactify p directions to circles; let these be the directions $x^1, \dots x^p$. This leaves $9-p$ space directions noncompact. Thus we will get a black hole in $(9-p)+1$ spacetime dimensions.

(ii) We wrap a string along one of the compact directions, say x^1. In the noncompact directions, we let this string be at the origin: $\{x^{p+1}, \dots x^9\} = 0$. Let the length of the x^1 circle be L. Then the energy of the wrapped string is

$$E = T_{NS1} L \tag{1.1}$$

From the viewpoint of the noncompact directions, this gives a point mass

$$m = T_{NS1} L \tag{1.2}$$

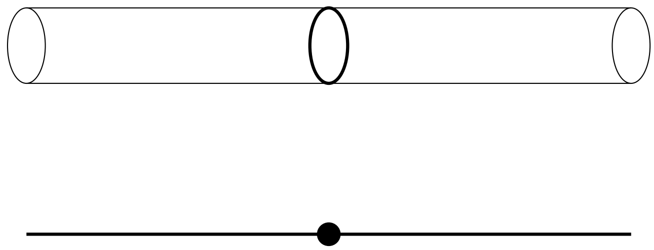

Figure 1.1: (a) The horizontal direction represents the noncompact space directions, while the vertical direction represents the compact directions. A string is wrapped on a compact circle. (b) From the viewpoint of the dimensionally reduced theory, we see only the noncompact directions. The string now appears as an object with some mass m.

(a) (b)

Figure 1.2: (a) If we take many separately wound strings, then we would be describing many separate masses. (b) We take one multiwound string; this is a bound state, and so corresponds to *one* massive object.

at the location $\{x^{p+1}, \ldots x^9\} = 0$ (fig.1.1).

(iii) We want a large amount of mass to make a big black hole. Thus we take n_1 strings wrapped as above, with

$$n_1 \gg 1 \tag{1.3}$$

(iv) If we have n_1 separately wrapped strings (as in fig.1.2(a)), then we would be trying to make n_1 separate tiny black holes. We wish to make *one* large black hole. Thus we need to consider a bound state of these n_1 strings. We have already seen that such a bound state has a simple form: the string wraps n_1 times around x^1 before closing. From the viewpoint of the noncompact directions, this gives an object of mass

$$m = n_1 T_{NS1} L \tag{1.4}$$

at the location $\{x^{p+1}, \ldots x^9\} = 0$ (fig.1.2(b)).

Now we ask: has this construction resulted in a black hole? To answer this question, we must compute the metric produced by the above mentioned strings.

1.1.1 The metric produced by strings

Let us take type IIA string theory for concreteness, and for the moment let all directions be noncompact. It will be convenient to begin by looking at the string metric $g^S_{\mu\nu}$, though later we will consider the Einstein metric $g^E_{\mu\nu}$ as well.

Let us separate a direction x^1 along which we will presently place our strings. Introducing polar coordinates in the directions transverse to x^1, the flat spacetime metric is

$$ds^2_S = -dt^2 + dx_1^2 + dr^2 + r^2 d\Omega_7^2 \tag{1.5}$$

Now consider n_1 string stretching along the direction x^1, placed at the location $r = 0$ in the transverse space. The metric produced by these strings has the form []

$$ds^2_S = H^{-1}[-dt^2 + dx_1^2] + dr^2 + r^2 d\Omega_7^2 \tag{1.6}$$

where

$$H = 1 + \frac{Q_1}{r^6} \tag{1.7}$$

Metrics of strings and branes will be crucial in our study, so let us analyze this metric in some detail:

(i) The string has a 2-dimensional worldsheet spanning the directions t, x_1. We see that the metric is Lorentz invariant in this t, x_1 space. This invariance reflects the fact that the action of the string is just given by the area of the worldsheet.

(ii) The coefficient function H is a harmonic function in the 7-dimensional space transverse to the brane:

$$\triangle H \equiv \sum_{i=2}^{8} \frac{\partial^2}{\partial x_i^2} H = 0 \tag{1.8}$$

This is analogous to the fact that in the Schwarzschild metric we get $g_{tt} = -(1 - \frac{2M}{r})$, which is harmonic in 3-dimensional space.

(iii) The quantity Q_1 is proportional to the number of strings n_1

$$Q_1 \propto n_1 \tag{1.9}$$

We assume that $n_1 \gg 1$ to get strong sources whose metric can be well described by a classical approximation.

The string is a charged object, and radiates the gauge field B_{ab}. The coupling of the string to the gauge field has the form of an integral along the worldsheet: $\int B d\mathcal{A}$. The worldsheet of the string lies along t, x_1. Thus we expect to generate a gauge field B_{tx_1}. We have

$$B_{tx_1} = H = 1 + \frac{Q_1}{r^6} \tag{1.10}$$

This gauge field is analogous to the field produced by a point charge q in electromagnetism. In the electromagnetic case the worldline is along t, and the coupling has the form: $q\int Adl$. This results in a gauge field component $A_t = \frac{q}{r}$, which is just the scalar potential produced by a stationary point charge. This scalar potential is harmonic: $\triangle A_t = 0$. Similarly, B_{tx_1} is harmonic in the transverse space $x^2, \dots x^9$. Note that we can add a constant to B_{tx_1} by a gauge transformation. Thus we can make $B_{tx_1} \to 0$ at infinity, and this potential then looks entirely analogous to the electromagnetic case.

The gravitational solution produced by the string has one more nontrivial component: the dilaton ϕ. Recall that from an M-theory perspective, the string is given by a M2 brane wrapped on the direction x^{11}. The tension of the M brane tends to squeeze this direction to a smaller value near the location of the branes. But the quantity e^ϕ reflects the size of the x^{11} direction. Thus we expect that e^ϕ will approach smaller values as we approach $r = 0$. We have

$$e^\phi = H^{-1} = (1 + \frac{Q_1}{r^6})^{-1} \tag{1.11}$$

We see that as $r \to 0$

$$e^\phi \approx \frac{r^6}{Q_1} \to 0 \tag{1.12}$$

so $\phi \to -\infty$ and the x^{11} circle gets pinched to zero length.

Let us summarize the gravitational solution produced by elementary strings lying along x_1

$$\begin{aligned} ds_S^2 &= H^{-1}[-dt^2 + dx_1^2] + dr^2 + r^2 d\Omega_7^2 \\ B_{tx_1} &= H^{-1} \\ e^{2\phi} &= H^{-1} \end{aligned} \tag{1.13}$$

with

$$H = 1 + \frac{Q_1}{r^6} \tag{1.14}$$

a harmonic function in the transverse 8-dimensional space. What is remarkable is that we can replace H by *any* harmonic function, and still get a solution to the field equations. Thus we can take

$$H = 1 + \frac{q_1}{|\vec{x} - \vec{x}_1|^6} + \frac{q_2}{|\vec{x} - \vec{x}_2|^6} + \dots + \frac{q_k}{|\vec{x} - \vec{x}_k|^6} \tag{1.15}$$

where $\vec{x}_1, \vec{x}_2, \dots \vec{x}_k$ are points in the 8-dimensional transverse space. This solution corresponds to string sources with strengths q_i at locations $\vec{x}_i$, with all strings stretching along the direction x_1.

1.1.2 Compactifying directions

In the above discussion we had 9+1 noncompact dimensions. Let us now see how we can compactify p of these directions and move towards the black hole solution we desire.

Compactifying x_1

The string above stretched for an infinite length along the x_1 direction. But a black hole should look like a localized object in spacetime. Thus the direction x_1 must be among the directions we compactify.

We see that the solution (1.13) is independent of x_1. Thus it remains a solution if we assume that

$$0 \le x_1 < L_1 \tag{1.16}$$

and that we have the identification

$$x_1 = 0 \quad \leftrightarrow \quad x_1 = L_1 \tag{1.17}$$

Thus compactification along the direction of the string does not change the algebraic form of the solution.

Compacitfying a transverse direction x_2

Now supposed we wish to compactify a transverse direction like x_2 on a circle of length L_2. A solution with such a compactification would be periodic under the shift

$$x_2 \ \rightarrow x_2 + L_2 \tag{1.18}$$

Unlike the situation with x_1, the solution (1.13) is not periodic under shifts of x_2. To get a solution with the required periodicity, we take a 1-dimensional array of string sources along the x_2 direction, placed at locations

$$x_2 = nL_2, \quad -\infty < n < \infty \tag{1.19}$$

Thus the harmonic function H has the form

$$H = 1 + \sum_{n=-\infty}^{\infty} \frac{Q_1}{(r'^2 + (x_2 - nL_2)^2)^3} \tag{1.20}$$

where

$$r'^2 = \sum_{i=3}^{9} x_i^2 \tag{1.21}$$

Bu construction, the functions appearing in the solution (1.13) are now periodic under (1.18), and we have compactified the transverse direction x_2.

We will actually be interested in a useful approximation of the above solution. We normally think of the compact directions as having a fixed, small size, perhaps of order planck length. The length scale set by Q_1, on the other hand will be large, since $Q_1 \propto n_1$, and we make a large black hole by taking $n_1 \gg 1$. Thus we wish to consider the limit

$$L_2 \ll Q_1^{\frac{1}{6}} \tag{1.22}$$

In this limit the sum in (1.20) can be replaced by an integral

$$\sum_{n=-\infty}^{\infty} \frac{Q_1}{(r'^2+(x_2-nL_2)^2)^3} \approx \int_{n=-\infty}^{\infty} \frac{Q_1}{(r'^2+(x_2-nL_2)^2)^3} = \frac{3\pi}{8L_2}\frac{1}{r'^5} \quad (1.23)$$

Thus H takes the form

$$H = 1 + \frac{Q_1'}{r'^5} \quad (1.24)$$

which is a harmonic function in the noncompact space $x^3, \dots x^9$, which is now 7-dimensional.

Proceeding in this way, we find that if we compactify x_1 and $p-1$ other directions, then the solution has the form (1.13) with

$$H = 1 + \frac{Q_1''}{r''^{(7-p)}} \quad (1.25)$$

where r'' is the radial coordinate in the noncompact directions.

We will be interested in the case where we compactify x_1 and 4 other directions. In that case the harmonic function has the form

$$H = 1 + \frac{Q_1''}{r''^2} \quad (1.26)$$

1.1.3 Looking for a horizon

In the above discussion we have worked with the string metric g^S_{ab}. The physics of black holes was, on the other hand, is naturally described in the Einstein metric g^E_{ab}. For example the entropy formula

$$S_{bek} = \frac{A}{4G} \quad (1.27)$$

has the area of the horizon A measured using the Einstein metric. Thus we start by converting the solution (1.13) to the Einstein metric.

We have

$$g^E_{ab} = e^{-\frac{\phi}{2}} g^S_{ab} \quad (1.28)$$

This gives

$$ds^2_E = H^{-\frac{3}{4}}[-dt^2 + dx_1^2] + H^{\frac{1}{4}}[dr^2 + r^2 d\Omega_3^2] + H^{\frac{1}{4}} \sum_{i=1}^{4} dx_i dx_i \quad (1.29)$$

To look for a horizon, we consider a surface of constant r, and take the limit $r \to 0$. A constant r surface is a cylindrical 8-dimensional surface, with the following structure:

(i) In the transverse directions, we have a S^3 of radius $H^{\frac{1}{8}}$. The area of this sphere is

$$A_{S^3} = (2\pi^2)H^{\frac{3}{8}} \tag{1.30}$$

where

$$A_{\Omega^3} = 2\pi^2 \tag{1.31}$$

is the volume of a 3-sphere of unit radius.

(ii) The direction x^1 has the length

$$L_{x^1} = H^{-\frac{3}{8}} L_1 \tag{1.32}$$

(iii) The directions along the T^4 have a volume

$$V_4 = H^{\frac{1}{2}} V_4 \tag{1.33}$$

Thus the overall area of the surface at radius $r = r_0$ is

$$A_H = \left((2\pi^2)H^{\frac{3}{8}} r^3\right)\left(H^{-\frac{3}{8}} L_1\right)\left(H^{\frac{1}{2}} V\right) = 2\pi^2 LVH^{\frac{1}{2}} r^3 \approx 2\pi^2 LVQ_1^{\frac{1}{2}} r^2 \tag{1.34}$$

We see that in the limit $r_0 \to 0$, we get

$$A_H \to 0 \tag{1.35}$$

and we have not succeeded in getting a black hole.

It is not hard to see the reason for our failure. The strings wrap around the direction x^1. Their tension pinches this circle, making it have zero length at the location of the strings. We will now see that our failure to get a nonzero A_H here is a *good* thing, since a nonzero A_H would have led to an immediate problem for string theory.

The entropy of the NS1 solution

Since $A_H = 0$, we find that the Bekenstein entropy is

$$S_{bek} = \frac{A_H}{4G} = 0 \tag{1.36}$$

Let us compare this to the *microscopic* entropy S_{micro} of our system. We have taken a bound state of n_1 strings, with no excitations on these strings. We have seen in (**??**) that the degeneracy of such a bound state is

$$\mathcal{N} = 256 \tag{1.37}$$

Thus

$$S_{micro} = \ln[256] \approx 0 \tag{1.38}$$

The symbol ≈ 0 here means that S_{micro} is a fixed number, rather than a number that grows with n_1. We take $n_1 \gg$ to make a good classical solution fro our string source. If A_H was nonzero for this classical solution, then it would be something that increased with n_1, and this would contradict (1.38). As things have turned out, we can write

$$S_{bek} = S_{micro} \approx 0 \tag{1.39}$$

and string theory has saved itself from a problem.

1.2 The NS1-P solution

The tension of the NS1 strings had pinched the circle x^1, making the horizon area zero. What should we do to make the x^1 direction not pinch?

To get a idea of what we need, consider the energy of the wrapped strings without, for the moment, considering their backreaction on the metric. The energy is

$$E_{NS1} = n_1 T_{NS1} L_1 \tag{1.40}$$

This energy is minimized for $L_1 = 0$, which is why the tension of the strings pinches the x^1 circle to zero at the location of the strings.

We thus need to add something whose energy will *increase* when L_1 is decreased. Consider a graviton carrying n_p units of momentum along the x^1. The energy of this momentum mode will be

$$E_P = \frac{2\pi n_p}{L_1} \tag{1.41}$$

This energy increases when L_1 is decreased. Assuming that the directions $x_1, \ldots x_4$ and y are compactified, one finds that the metric of such a graviton mode is

$$ds^2_{string} = H^{-1}[-dt^2 + dx_1^2 + K(dt + dx_1)^2] + [dr^2 + r^2 d\Omega_3^2] + \sum_{i=1}^{4} dx_i dx_i \tag{1.42}$$

where

$$K = \frac{Q_p}{r^2} \tag{1.43}$$

and the gauge field and dilaton vanish

$$B_{\mu\nu} = 0, \quad e^{2\phi} = 1 \tag{1.44}$$

The quantity Q_p is proportional to the number of units of momentum

$$Q_p = \frac{g^2 \alpha'^4}{V R^2} n_p \tag{1.45}$$

Now we take both NS1 and P charges. The solution is

$$\begin{aligned} ds^2_{string} &= H_1^{-1}\left[(-dt^2+dx_1^2)+K(dt+dx_1)^2\right]+[dr^2+r^2d\Omega_3^2]+\sum_{i=1}^{4}dx_idx_i \\ B_{tx_1} &= H^{-1} \\ e^{2\phi} &= H^{-1} \end{aligned} \tag{1.46}$$

1.2.1 Microscopic entropy of the NS1P bound state

We have a bound state of n_1 strings and n_p units of momentum. We have seen that this bound state is degenerate: there are

$$\mathcal{N} \approx e^{2\sqrt{2}\pi\sqrt{n_1 n_p}} \tag{1.47}$$

states with the same mass and charge. These different states correspond to different ways in which the momentum n_p can be carried along the multiwound string as travelling waves. Thus the microscopic entropy is

$$S_{micro} = \ln \mathcal{N} = 2\sqrt{2}\pi\sqrt{n_1 n_p} \tag{1.48}$$

This time we have an entropy that increases with the charges n_1, n_p. Let us now compute the area of the horizon.

1.3 The entropy of a gas of vibrations

Let a direction y be compactified to a circle of length L. Consider a string wrapped n_w on this circle; thus the total length of the string is $L_T = n_w L$. Let the string carry n_p units of momentum along this string, say in the positive y direction. At this moment we put no excitations traveling in the negative y direction, though we will do that as well later.

The total energy and momentum on the string must have the form

$$E = P = \frac{2\pi n_p}{L} = \frac{2\pi n_w n_p}{L_T} \tag{1.49}$$

The individual excitations carrying thus momentum can be in various harmonics k on the string with length L_T. . An excitation in the kth harmonic has energy and momentum

$$e_k = p_k = \frac{2\pi k}{L_T} \tag{1.50}$$

Let there be m_k excitations in the harmonic k. Then we must have

$$\sum_{k=1}^{\infty} k\, m_k = n_w n_p \tag{1.51}$$

Our goal is to find the number $\mathcal{N}$ of sets $\{n_k\}$ which satisfy this relation; this number $\mathcal{N}$ gives the degeneracy of states with the quantum numbers (1.49), and the entropy of the vibrating string will then be given by

$$S_{micro} = \ln \mathcal{N} \tag{1.52}$$

1.3.1 The partition function

The partition function of a system is defined as

$$Z[\beta] = \sum_{states} e^{-\beta E_{state}} \tag{1.53}$$

where

$$\beta = \frac{1}{T} \tag{1.54}$$

and T is the temperature. We first consider the partition function for a single boson, then for a single fermion, and finally, for the case where we have f_B 'flavors' of bosons and f_F 'flavors' of fermions.

The partition function for a single boson

Consider just one bosonic fourier mode k. Each excitation of this mode has energy $e_k = 2\pi k/L_T$. Since the number of excitations can be $m_k = 0, 1, 2, \dots$, summing over various numbers of these excitations gives the contribution

$$Z_k^B = \sum_{m_k=0}^{\infty} e^{-\beta m_k e_k} = \frac{1}{1 - e^{-\beta e_k}} \tag{1.55}$$

Since the various k harmonics describe independent sets of excitations, the corresponding contributions Z_k need to be multiplied together. We can consider the log of Z, where we find

$$\ln Z^B = \sum_{k=1}^{\infty} \ln Z_K^B = -\sum_{k=1}^{\infty} \ln[1 - e^{-\beta e_k}] \tag{1.56}$$

For large values of n_w, n_p, the values of k are peaked at $k \gg 1$, where we can approximate the sum over k by an integral

$$\ln Z^B \to -\int_0^{\infty} dk \ln[1 - e^{-\beta e_k}] = -\frac{L_T}{2\pi} \int_0^{\infty} de_k \ln[1 - e^{-\beta e_k}] = \frac{L_T}{2\pi\beta} \frac{\pi^2}{6} \tag{1.57}$$

The partition function for a single fermions

A fermion mode can have only two possible occupation numbers $m_k = 0, 1$. Thus in place of (1.55) we get

$$Z_k^F = \sum_{m_k=0}^{1} e^{-\beta m_k e_k} = 1 + e^{-\beta e_k} \tag{1.58}$$

Again taking the log of Z^F and approximating the sum over k by an integral, we get

$$\log Z^F \to \frac{L_T}{2\pi}\int_0^\infty de_k \ln[1+e^{-\beta e_k}] = \frac{L_T}{2\pi\beta}\frac{\pi^2}{12} \tag{1.59}$$

We see that $\ln Z^F = \frac{1}{2}\ln Z^B$, so a fermions counts as 'half a boson' for the purposes of its contribution to the partition function.

Several flavors of bosons and fermions

Suppose we have f_B bosonic degrees of freedom and f_F fermionic degrees of freedom. Since each degree of freedom gives independent excitations, the corresponding partition functions Z are multiplied together, which leads to a sum over the corresponding logarithms

$$\log Z = f_B \log Z^B + f_F \log f^F = (f_B + \frac{1}{2}f_F)\frac{\pi L_T}{12\beta} \equiv c(\frac{\pi L_T}{12\beta}) \tag{1.60}$$

where we have defined

$$c = f_B + \frac{1}{2}f_F \tag{1.61}$$

The quantity c is called the 'central charge' and gives a measure of the effective degrees of freedom of the system.

1.3.2 The thermodynamics of string vibrations

Let us now use the above partition function to compute various thermodynamics quantities describing the vibrating string.

General relations

The average energy is given in terms of the partition function by

$$E = \frac{1}{Z}(-\partial_\beta)Z = -\partial_\beta \ln Z \tag{1.62}$$

Applying this to (1.60), we find

$$E = \frac{c\pi L_T}{12\beta^2} \tag{1.63}$$

Thus if we put an energy E on the string, and assume that it is distributed thermally among all possible excitations, then this thermal distribution will be characterized by a temperature

$$T = \beta^{-1} = \sqrt{\frac{12E}{\pi L_T c}} \tag{1.64}$$

The entropy of the distribution is $S = \ln \mathcal{N}$, where $\mathcal{N}$ is the average number of states that contribute to the partition function. Thus we can write

$$Z = \sum_{states} e^{-\beta E_{state}} \sim e^{S-\beta E} \tag{1.65}$$

This gives

$$S = \ln Z + \beta E \tag{1.66}$$

Applying this to (1.60), we find

$$S = \frac{c\pi L_T}{6\beta} = \sqrt{\frac{c\pi L_T E}{3}} \tag{1.67}$$

1.3.3 The string with charges n_w, n_p

Let us now apply these general relations to our case of the 2-charge extremal system–the string with winding n_w and momentum n_p.

Thermodynamic quantities

The string has 8 transverse directions in which it can vibrate, so $f_B = 8$. By supersymmetry, there are a corresponding number of fermionic flavors, so $f_F = 8$. Thus

$$c = f_B + \frac{1}{2} f_F = 8 + 4 = 12 \tag{1.68}$$

We have noted that

$$E = \frac{2\pi n_w n_p}{L_T}, \quad L_T = n_w L \tag{1.69}$$

Then (1.64) gives

$$T = \frac{2}{L}\sqrt{\frac{n_p}{n_w}} \tag{1.70}$$

and (1.67) gives

$$S = 2\sqrt{2}\pi\sqrt{n_1 n_p} \tag{1.71}$$

Qualitiative picture of the vibrating string

From the above computation we can extract a few other details. In a thermal distribution, the average energy of an excitation is

$$\bar{e} \sim T \sim \frac{\sqrt{n_1 n_p}}{L_T} \tag{1.72}$$

From (1.50) we see then that that the generic quantum is in a harmonic

$$\bar{k} \sim \sqrt{n_1 n_p} \tag{1.73}$$

on the multiwound string. Given that the total energy is (1.69), we find that the number of quanta is

$$\bar{m} \sim \sqrt{n_1 n_p} \tag{1.74}$$

From (1.55) we can find the average occupation number of a bosonic energy level e_k

$$\langle m_k \rangle = -\frac{1}{Z_k^B}\frac{1}{e_k}\partial_\beta Z_k^B = \frac{1}{e^{\beta e_k}-1} \tag{1.75}$$

so for the generic quantum with $e_k \sim \beta^{-1}$ we have

$$\langle m_k \rangle \sim 1 \tag{1.76}$$

For fermionic levels,

$$\langle m_k \rangle = -\frac{1}{Z_k^F}\frac{1}{e_k}\partial_\beta Z_k^F = \frac{1}{e^{\beta e_k}+1} \tag{1.77}$$

so for the generic quantum with $e_k \sim \beta^{-1}$ we again have

$$\langle m_k \rangle \sim 1 \tag{1.78}$$

To summarize, there are a large number of ways to partition the energy into different harmonics. One extreme possibility is to put all the energy into the lowest harmonic $k = 1$; then the occupation number of this harmonic will be

$$m = n_1 n_p \tag{1.79}$$

At the other extreme we can put all the energy into a single quantum in the harmonic $n_1 n_p$; i.e.

$$k = n_1 n_p, \quad m_k = 1 \tag{1.80}$$

But the *generic* state which contributes to the entropy has its typical excitations in harmonics with $k \sim \sqrt{n_1 n_p}$. There are $\sim \sqrt{n_1 n_p}$ such harmonic modes; and the occupation number of each such mode is $< m_k > \sim 1$. These details about the generic state will be important to us later.

1.3.4 The Bekenstein entropy of the NS1P state

The Einstein metric is

$$ds_E^2 = H^{-\frac{3}{4}}[-dt^2+dx_1^2+K(dt+dx_1)^2]+H^{\frac{1}{4}}[dr^2+r^2 d\Omega_3^2]+H^{\frac{1}{4}}\sum_{i=1}^{4} dx_i dx_i \tag{1.81}$$

In the limit $r \to 0$, we get

$$ds^E \to \frac{r^{\frac{3}{2}}}{Q_1^{\frac{3}{4}}}[-dt^2+dx_1^2]+\frac{Q_p}{r^{\frac{1}{2}}Q_1^{\frac{3}{4}}}(dt+dx_1)^2+\frac{Q_1^{\frac{1}{4}}}{r^{\frac{1}{2}}}dr^2+Q_1^{\frac{1}{4}}r^{\frac{3}{2}}d\Omega_3^2+\frac{Q_1^{\frac{1}{4}}}{r^{\frac{1}{2}}}\sum_{i=1}^{4} dx_i dx_i \tag{1.82}$$

Now look at the hypersurface $t = 0$. The surface at a given value of r has the following components to its area:

(i) The direction along x_1 has a length

$$L_{x_1} = \frac{Q_p^{\frac{1}{2}}}{r^{\frac{1}{4}} Q_1^{\frac{3}{8}}} L \tag{1.83}$$

(ii) The directions along the torus give

$$V_4 = \frac{Q_1^{\frac{1}{2}}}{r} V \tag{1.84}$$

(iii) The angular sphere gives

$$V_{\Omega_3} = 2\pi^2 Q_1^{\frac{3}{8}} r^{\frac{9}{4}} \tag{1.85}$$

Thus the area of the horizon is

$$A_H = \left(\frac{Q_p^{\frac{1}{2}}}{r^{\frac{1}{4}} Q_1^{\frac{3}{8}}} L \right) \left(\frac{Q_1^{\frac{1}{2}}}{r} V \right) \left(2\pi^2 Q_1^{\frac{3}{8}} r^{\frac{9}{4}} \right) = 2\pi^2 L V Q_p^{\frac{1}{2}} Q_1^{\frac{1}{2}} r \tag{1.86}$$

Once again, we find that as we take $r \to 0$, we get

$$A_H \to 0 \tag{1.87}$$

So we still do not seem to have a black hole.

This time however, there is a difference: we do have, in fact a nonzero horizon area; its just that in our present approxmation we are not able to see it.

1.3.5 The NS1-NS5-P black hole

The metric produced by such NS5 branes is

$$ds^2_{string} = [-dt^2 + dy^2] + H_5[dr^2 + r^2 d\Omega_3^2] + H_5 dx_i dx_i \tag{1.88}$$

where

$$H_5 = 1 + \frac{Q_5}{r^2} \tag{1.89}$$

Here Q_5 is proportional to the number n_5 of NS5 branes

$$Q_5 = \alpha' n_5 \tag{1.90}$$

We get a dilaton

$$e^{2\phi} = H_5 \tag{1.91}$$

We also get a $B_{\mu\nu}$ field. Recall that the NS5 brane is the magnetic dual of the NS1 brane. Thus the $B_{\mu\nu}$ field lies along directions of the angular sphere S^3

$$B_{\phi\psi} = \sin^2\theta \tag{1.92}$$

We can now write down the solution with NS1-NS5-P charges

$$\begin{aligned} ds^2_{string} &= H_1^{-1}\left[(-dt^2 + dx_1^2) + K(dt + dx_1)^2\right] + H_5[dr^2 + r^2 d\Omega_3^2] + \sum_{i=1}^{4} dx_i dx_i \\ B_{tx_1} &= H_1^{-1}, \quad B_{\phi\psi} = \sin^2\theta \\ e^{2\phi} &= \frac{H_5}{H_1} \end{aligned} \tag{1.93}$$

The horizon area has the following components

(i) The angular directions have the area

$$A_{\omega_3} = 2\pi^2 H_5^{\frac{3}{2}} r^3 \tag{1.94}$$

(ii) The torus directions have the volume

$$A_{T^4} = V \tag{1.95}$$

(iii) The x_1 direction has a length dominated by the term

$$L_{x_1} = \frac{K^{\frac{1}{2}}}{H_1^{\frac{1}{2}}} L \tag{1.96}$$

Thus the overall area of the horizon in the string metric is

$$A_H^{string} = \left(2\pi^2 H_5^{\frac{3}{2}} r^3\right)(V)\left(\frac{K^{\frac{1}{2}}}{H_1^{\frac{1}{2}}} L\right) = 2\pi^2 V L r^3 H_5^{\frac{3}{2}} H_1^{-\frac{1}{2}} K^{\frac{1}{2}} \tag{1.97}$$

The 10-D Einstein metric $g^E_{\mu\nu}$ is related to the string metric $g^S_{\mu\nu}$ by

$$g^E_{\mu\nu} = e^{-\frac{\phi}{2}} g^S_{\mu\nu} = \left(\frac{H_1}{H_5}\right)^{\frac{1}{4}} \tag{1.98}$$

Thus

$$A^E = \left(\frac{g^E_{\mu\nu}}{g^S_{\mu\nu}}\right)^4 = \frac{H_1}{H_5} A^{string} = 2\pi^2 V L r^3 H_5^{\frac{1}{2}} H_1^{\frac{1}{2}} K^{\frac{1}{2}} \to 2\pi^2 V L \sqrt{Q_1 Q_5 Q_p} \tag{1.99}$$

Recalling that

$$Q_1 = \frac{g^2\alpha'^3}{V}n_1, \quad Q_5 = \alpha' n_5, \quad Q_p = \frac{g^2\alpha'^4}{VR^2}n_p \tag{1.100}$$

and that

$$G = 8\pi^6 g^2\alpha'^4 \tag{1.101}$$

we find that

$$S_{bek} = \frac{A_H}{4G} = 2\pi\sqrt{n_1 n_5 n_p} \tag{1.102}$$

The nonextremal gravity solution

We continue to use the compactification $M_{9,1} \to M_{4,1} \times T^4 \times S^1$. We have charges NS1, NS5, P as before, but also extra energy that gives nonextremality. The metric and dilaton are [**?**]

$$ds^2_{string} = H_1^{-1}[-dt^2+dy^2+\frac{r_0^2}{r^2}(\cosh\sigma dt+\sinh\sigma dy)^2]+H_5[\frac{dr^2}{(1-\frac{r_0^2}{r^2})}+r^2 d\Omega_3^2]+\sum_{a=1}^{4} dz_a dz_a \tag{1.103}$$

$$e^{2\phi} = \frac{H_5}{H_1} \tag{1.104}$$

where

$$H_1 = 1 + \frac{r_0^2\sinh^2\alpha}{r^2}, \quad H_5 = 1 + \frac{r_0^2\sinh^2\gamma}{r^2} \tag{1.105}$$

The integer valued charges carried by this hole are

$$\hat{n}_1 = \frac{Vr_0^2\sinh 2\alpha}{2g^2\alpha'^3} \tag{1.106}$$

$$\hat{n}_5 = \frac{r_0^2\sinh 2\gamma}{2\alpha'} \tag{1.107}$$

$$\hat{n}_p = \frac{R^2Vr_0^2\sinh 2\sigma}{2g^2\alpha'^4} \tag{1.108}$$

The energy (i.e. the mass of the black hole) is

$$E = \frac{RVr_0^2}{2g^2\alpha'^4}(\cosh 2\alpha + \cosh 2\gamma + \cosh 2\sigma) \tag{1.109}$$

The horizon is at $r = r_0$. From the area of this horizon we find the Bekenstein entropy

$$S_{Bek} = \frac{A_{10}}{4G_{10}} = \frac{2\pi RVr_0^3}{g^2\alpha'^4}\cosh\alpha\cosh\gamma\cosh\sigma \tag{1.110}$$

The Hawking temperature is

$$T_H = [(\frac{\partial S}{\partial E})_{\hat{n}_1,\hat{n}_5,\hat{n}_p}]^{-1} = \frac{1}{2\pi r_0\cosh\alpha\cosh\gamma\cosh\sigma} \tag{1.111}$$

The extremal limit: 'Three large charges, no nonextremality'

The extremal limit is obtained by taking

$$r_0 \to 0, \quad \alpha \to \infty, \quad \gamma \to \infty, \quad \sigma \to \infty \tag{1.112}$$

while holding fixed

$$r_0^2 \sinh^2 \alpha = Q_1, \quad r_0^2 \sinh^2 \gamma = Q_5, \quad r_0^2 \sinh^2 \sigma = Q_p \tag{1.113}$$

This gives the extremal hole we constructed earlier. For this case we have already checked that the microscopic entropy agrees with the Bekenstein entropy (**??**). It can be seen that in this limit the Hawking temperature is $T_H = 0$.

Two large charges + nonextremality

We now wish to move away from the extremal 3-charge system, towards the neutral Schwarzschild hole. For a first step, we keep two of the charges large; let these be NS1, NS5. We will have a small amount of the third charge P, and a small amount of nonextremality. The relevant limits are

$$r_0, \ r_0 e^{\sigma} \ \ll r_0 e^{\alpha}, \ r_0 e^{\gamma} \tag{1.114}$$

Thus σ is finite but $\alpha, \gamma \gg 1$. We are 'close' to the extremal NS1-NS5 state, so we can hope that the excitations will be a small correction. The excitations will be a 'dilute' gas among the large number of $\hat{n}_1, \hat{n}_5$ charges and a simple model for these excitations might give us the entropy and dynamics of the system.

The BPS mass corresponding to the $\hat{n}_1$ NS1 branes is

$$M_1^{BPS} = \frac{R\hat{n}_1}{\alpha'} = \frac{RVr_0^2}{2g^2\alpha'^4}\sinh 2\alpha = \frac{RVr_0^2}{2g^2\alpha'^4}(\cosh 2\alpha - e^{-2\alpha}) \approx \frac{RVr_0^2}{2g^2\alpha'^4}\cosh 2\alpha \tag{1.115}$$

The BPS mass corresponding to the $\hat{n}_5$ NS5 branes is

$$M_5^{BPS} = \frac{RV\hat{n}_5}{g^2\alpha'^3} = \frac{RVr_0^2}{2g^2\alpha'^4}\sinh 2\gamma = \frac{RVr_0^2}{2g^2\alpha'^4}(\cosh 2\gamma - e^{-2\gamma}) \approx \frac{RVr_0^2}{2g^2\alpha'^4}\cosh 2\gamma \tag{1.116}$$

Thus the energy (1.109) can be written as

$$E = M_1^{BPS} + M_5^{BPS} + \Delta E, \quad \Delta E \approx \frac{RVr_0^2}{2g^2\alpha'^4}\cosh 2\sigma \tag{1.117}$$

The momentum is

$$P = \frac{\hat{n}_p}{R} = \frac{RVr_0^2}{2g^2\alpha'^4}\sinh 2\sigma \tag{1.118}$$

Note that

$$\Delta E + P \approx \frac{RVr_0^2}{2g^2\alpha'^4}e^{2\sigma}, \quad \Delta E - P \approx \frac{RVr_0^2}{2g^2\alpha'^4}e^{-2\sigma} \tag{1.119}$$

We wish to compute the entropy (1.110) in this limit. Note that

$$\hat{n}_1 = \frac{V r_0^2}{2g^2\alpha'^3}\sinh 2\alpha \approx \frac{V r_0^2}{g^2\alpha'^3}\cosh^2\alpha \qquad (1.120)$$

$$\hat{n}_5 = \frac{r_0^2}{2\alpha'}\sinh 2\gamma \approx \frac{r_0^2}{\alpha'}\cosh^2\gamma \qquad (1.121)$$

We then find

$$S_{Bek} \approx 2\pi\sqrt{\hat{n}_1\hat{n}_5}\ [\ \sqrt{\frac{R}{2}(\Delta E + P)} + \sqrt{\frac{R}{2}(\Delta E - P)}\ ] \qquad (1.122)$$

Let us now look at the microscopic description of this nonextremal state. The NS1, NS5 branes generate an 'effective string' as before. In the extremal case all the excitations were right movers (R) on this effective string, so that we had the maximal possible momentum charge P for the given energy. For the non-extremal case we will have momentum modes moving in both R,L directions. Let the right movers carry n_p units of momentum and the left movers $\bar{n}_p$ units of (oppositely directed) momentum. Then (ignoring any interaction between the R,L modes) we will have

$$\Delta E = \frac{1}{R}(n_p + \bar{n}_p), \quad P = \frac{1}{R}(n_p - \bar{n}_p) \qquad (1.123)$$

Since we have ignored any interactions between the R,L modes the entropy S_{micro} of this 'gas' of momentum modes will be the sum of the entropies of the R,L excitations. Thus using (**??**) we write

$$S_{micro} = 2\pi\sqrt{\hat{n}_1\hat{n}_5 n_p} + 2\pi\sqrt{\hat{n}_1\hat{n}_5\bar{n}_p} \qquad (1.124)$$

But using (1.123) in (1.122) we find

$$S_{micro} = 2\pi\sqrt{\hat{n}_1\hat{n}_5}\ [\ \sqrt{\frac{R}{2}(\Delta E + P)} + \sqrt{\frac{R}{2}(\Delta E - P)}\ ] \qquad (1.125)$$

Comparing to (1.122) we find that

$$S_{micro} \approx S_{Bek} \qquad (1.126)$$

We thus see that a simple model of the microscopic brane bound state describes well the entropy of this near extremal system.

Bibliography