Appendices

Appendix A

General Relativity: Notations and useful tools

In this appendix we collect together some basic relations in general relativity.

A.1 Notation

The metric is a symmetric tensor $g_{\mu\nu}$, transforming under a coordinate change as

$$g'_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \tag{A.1}$$

In a local orthonormal frame the metric take the form $\{-1, 1, 1, \ldots 1\}$. The inverse metric $g^{\mu\nu}$ is defined through

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu} \tag{A.2}$$

where $\delta^{\mu}_{\nu} = 1$ if $\mu = \nu$ and 0 otherwise. The determinant of $g_{\mu\nu}$ is called g. Volumes are defined using the Levi-Civita symbol, which in a local orthonormal frame has the value

$$\epsilon_{012\ldots(D-1)} = 1 \tag{A.3}$$

For any permutation of these indices, the value is given by the sign of the permutation. If the indices are not all different, the symbol is 0. In a general coordinate system x^{μ}, we get

$$\epsilon_{012\ldots(D-1)} = \sqrt{-g} \tag{A.4}$$

with the value for other index choices given by the same rules as above.

The connection is defined as

$$\gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\lambda\kappa} \left(g_{\kappa\nu,\lambda} + g_{\kappa\lambda,\nu} - g_{\nu\lambda,\kappa} \right) \tag{A.5}$$

The Riemann curvature tensor is

$$R^{\mu}{}_{\nu\lambda\kappa} = \Gamma^{\mu}_{\nu\kappa,\lambda} - \Gamma^{\mu}_{\nu\lambda,\kappa} + \Gamma^{\mu}_{\lambda\theta} \Gamma^{\theta}_{\nu\kappa} - \Gamma^{\mu}_{\kappa\theta} \Gamma^{\theta}_{\nu\lambda} \tag{A.6}$$

The Ricci tensor is

$$R_{\mu\nu} = R^{\theta}{}_{\mu\theta\nu} \tag{A.7}$$

The Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} \tag{A.8}$$

The Einstein action is

$$S_E = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}R \tag{A.9}$$

Here $d^Dx\sqrt{-g}$ gives the proper volume associated to the coordinate volume d^Dx. The variation of $\sqrt{-g}$ is given by

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.10}$$

The total action is given by adding the gravity action and the matter action

$$S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}R + S_{matter} \tag{A.11}$$

The stress tensor, also called the energy-momentum tensor, is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{A.12}$$

Requiring that the action (A.11) be stationary under variations of $g^{\mu\nu}$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{A.13}$$

gives the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{A.14}$$

A.2 Dimensional reduction

Suppose we start with the Einstein action, and compactify some of the space directions. Let us examine the form of the action that results upon this dimensional reduction.

Let the total spacetime have dimension D, and let the indices $A, B\dots$ run over $0, 1, \dots D-1$. Let p of the directions be compactified on circles; the coordinates in these directions are $y^i, i = 1, \dots p$. Let the coordinates in the remaining $D-p$ dimensions be $x^a, a = 0, 1, \dots D-p-1$. We assume that the metric g_{AB} is independent of the y^i, and further assume for the moment that

$$g_{ai} = 0 \tag{A.15}$$

We look for solutions where the metric of the compact directions has the form

$$g_{ij} = e^{C(x)}\delta_{ij} \tag{A.16}$$

Then we get

$$\Gamma^i_{ja} = \frac{1}{2}\delta^i_j C_{,a}, \quad \Gamma^a_{ij} = -\frac{1}{2}e^C g^{ab}C_{,b}\delta_{ij} \tag{A.17}$$

To study the Einstein action, we need to compute the curvature tensor in D dimensions and express it in terms of the curvature tensor in the $D-p$ dimensional space spanned by the x^a. Let us work around a point x in these $D-p$ directions. We can choose a system of coordinates around this point x so that at this poiny

$$g_{ab} = \eta_{ab}, \quad g_{ab,c} = 0 \tag{A.18}$$

We now express different components of the D dimensional Riemann tensor in terms of quantities in the $D-p$ dimensional spacetime spanned by the x^a. We find

$$[R^c{}_{adb}]_D = [R^c{}_{adb}]_{D-p} \tag{A.19}$$

Here the components $[R^c{}_{adb}]_D$ are computed using the metric g_{AB} in the entire D dimensional spacetime while $[R^c{}_{adb}]_{D-p}$ is computed by assuming that we are in a $D-p$ dimensional spacetime with the metric g_{ab}. We also get

$$\begin{aligned}
[R^i{}_{ajb}]_D &= \Gamma^i_{ab,j} - \Gamma^i_{aj,b} + \Gamma^i_{Aj}\Gamma^A_{ab} - \Gamma^i_{Ab}\Gamma^A_{aj} \\
&= -\Gamma^i_{aj,b} + \Gamma^i_{cj}\Gamma^c_{ab} - \Gamma^i_{kb}\Gamma^k_{aj} \\
&= -\Gamma^i_{aj,b} - \Gamma^i_{kb}\Gamma^k_{aj} \\
&= -\frac{1}{2}\delta^i_j C_{;ab} - \frac{1}{4}\delta^i_j C_{,a}C_{,b}
\end{aligned} \tag{A.20}$$

Here we have converted ordinary partial derivatives of C to covariant derivatives: we have vanishing connection on the $D-p$ dimensional spacetime due to the coordinate choice (A.37) , and when we go to general coordinates on this space the term C_{ab} becomes $C_{;ab}$. Similarly we get

$$[R^a{}_{ibj}]_D = -e^C[\frac{1}{2}\delta_{ij}C^{;a}{}_b + \frac{1}{4}\delta_{ij}C^{;a}C_{,b}] \tag{A.21}$$

$$\begin{aligned}
[R^i_{kjl}]_D &= \Gamma^i_{Aj}\Gamma^A_{kl} - \Gamma^i_{Al}\Gamma^A_{kj} \\
&= \Gamma^i_{cj}\Gamma^c_{kl} - \Gamma^i_{cl}\Gamma^c_{kj} \\
&= -\frac{1}{4}e^C C_{,c}C^{;c}[\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}]
\end{aligned} \tag{A.22}$$

Thus

$$[R_{ab}]_D = [R^c{}_{acb}]_D + [R^i{}_{aib}]_D = [R_{ab}]_{D-p} - \frac{p}{2}C_{;ab} - \frac{p}{4}C_{,a}C_{,b} \tag{A.23}$$

$$\begin{aligned}
[R_{ij}]_D &= [R^k{}_{ikj}]_D + [R^c{}_{icj}]_D = -\delta_{ij}\frac{(p-1)}{4}e^C C_{,c}C^{;c} - e^C[\frac{1}{2}\delta_{ij}C^{;c}{}_c + \frac{1}{4}\delta_{ij}C^{;c}C_{,c}] \\
&= -\delta_{ij}[\frac{p}{4}e^C C_{,c}C^{;c} + \frac{1}{2}e^C C^{;c}{}_c]
\end{aligned} \tag{A.24}$$

$$[R]_D = [R]_{D-p} - \frac{p}{2}C_{;c}{}^{c} - \frac{p}{4}C_{,c}C^{;c} - \frac{p^2}{4}C_{,c}C^{;c} - \frac{p}{2}C^{;c}{}_{c} = [R]_{D-p} - pC_{;c}{}^{c} - \frac{p(p+1)}{4}C_{,c}C^{;c} \quad (A.25)$$

Now consider the Einstein action of the D dimensional gravity theory

$$S_D = \frac{1}{16\pi G}\int d^D\xi\sqrt{-g_D}[R]_D \quad (A.26)$$

From (A.16) we see that

$$\sqrt{-g_D} = \sqrt{-g_{D-p}}e^{\frac{p}{2}C} \quad (A.27)$$

We then get

$$S_D = \frac{V}{16\pi G}\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}([R]_{D-p} - pC_{;c}{}^{c} - \frac{p(p+1)}{4}C_{,c}C^{;c}) \quad (A.28)$$

where

$$V = \int dy^1 \ldots dy^p \quad (A.29)$$

is the 'coordinate volume' of the compact directions. V is a constant; the variation in the scale of the compact directions is given through the function $C(x)$. The middle term in the bracket in (A.28) can be integrated by parts, and we get

$$S_D = \frac{V}{16\pi G}\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}([R]_{D-p} + \frac{p(p-1)}{4}C_{,c}C^{;c}) \quad (A.30)$$

Let us note in particular the expression for $p = 1$; i.e., the case where we compactify one direction. Then (A.30) gives

$$S_D = \frac{V}{16\pi G}\int d^{D-1}x\sqrt{-g_{D-1}}e^{\frac{1}{2}C}[R]_{D-1} \quad (A.31)$$

While the term $C_{,c}C^{;c}$ is absent, we still have a very nonlinear coupling between the metric and C, due to the factor $e^{\frac{1}{2}C}$ multiplying the curvature scalar. We will now see how redefining the metric will remove such a coupling.

A.3 Scaling the metric

The part $\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}[R]_{D-p}$ in (A.30) would look like the action for gravity in $D-p$ dimensions, if the factor $e^{\frac{p}{2}C}$ were absent. Our goal is to get rid of this factor by absorbing it in the metric g_{ab}. But this needs some care since $[R]_{D-p}$ contains not only g_{ab} but also derivatives of g_{ab}. Let us now perform the relevant steps.

Suppose we have a D dimensional spacetime $x^0, x^1, \ldots x^{D-1}$ with a metric g_{ab}. Suppose we also have a scalar function $\mu(x)$ on this spacetime. We can then define another 'metric tensor' q_{ab}

$$q_{ab} = e^{\mu}g_{ab} \quad (A.32)$$

Quantities computed using the metric q will be labeled with the subscript q, and quantities computed using the metric g will be labeled with a subscript g.

Using the new metric q we find

$$[\Gamma^a_{bc}]_q = [\Gamma^a_{bc}]_g + \frac{1}{2}[\delta^a_b \mu_{,c} + \delta^a_c \mu_{,b} - q_{bc}\mu^{,a}] \tag{A.33}$$

where in the last term the metric is raised by q^{ab}.

Working around a point x, choose local coordinates such that

$$q_{ab}(x) = \eta_{ab}, \quad q_{ab,c}(x) = 0 \tag{A.34}$$

We then have

$$[R^c{}_{adb}]_q = [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_q \tag{A.35}$$

$$\begin{aligned} [R^c{}_{adb}]_g &= [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_g + [\Gamma^c_{fd}\Gamma^f_{ab} - \Gamma^c_{fb}\Gamma^f_{ad}]_g \\ &= [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_q - \frac{1}{2}[\delta^c_a \mu_{,b} + \delta^c_b \mu_{,a} - q_{ab}\mu^{,c}]_{,d} \\ &+ \frac{1}{2}[\delta^c_a \mu_{,d} + \delta^c_d \mu_{,a} - q_{ad}\mu^{,c}]_{,b} \\ &+ \frac{1}{4}[\delta^c_f \mu_{,d} + \delta^c_d \mu_{,f} - q_{fd}\mu^{,c}][\delta^f_a \mu_{,b} + \delta^f_b \mu_{,a} - q_{ab}\mu^{,f}] \\ &- \frac{1}{4}[\delta^c_f \mu_{,b} + \delta^c_b \mu_{,f} - q_{fb}\mu^{,c}][\delta^f_a \mu_{,d} + \delta^f_d \mu_{,a} - q_{ad}\mu^{,f}] \end{aligned} \tag{A.36}$$

(Indices in the second, third, fourth lines are raised by q^{ab}.)

We now assume that we are in a coordinate system where

$$q_{ab} = \eta_{ab}, \quad q_{ab,c} = 0 \tag{A.37}$$

We then find that

$$\begin{aligned} [R^c{}_{adb}]_g &= [R^c{}_{adb}]_q - \frac{1}{2}[\delta^c_b \mu_{,ad} - q_{ab}\mu^{,c}{}_d] + \frac{1}{2}[\delta^c_d \mu_{,ab} - q_{ad}\mu^{,c}{}_b] \\ &- \frac{1}{4}[\delta^c_d q_{ab} - \delta^c_b q_{ad}]\mu_{,f}\mu^{,f} + \frac{1}{4}[\delta^c_d \mu_{,a}\mu_{,b} - \delta^c_b \mu_{,a}\mu_{,d}] \\ &- \frac{1}{4}[q_{ad}\mu^{,c}\mu_{,b} - q_{ab}\mu^{,c}\mu_{,d}] \end{aligned} \tag{A.38}$$

This yields

$$[R_{ab}]_g = [R_{ab}]_q + \frac{1}{2}(D-2)\mu_{;ab} + \frac{1}{2}q_{ab}\mu_{;c}{}^c - \frac{1}{4}(D-2)q_{ab}\mu_{,c}\mu^{,c} + \frac{1}{4}(D-2)\mu_{,a}\mu_{,b} \tag{A.39}$$

$$[R]_g = e^{\mu}[R]_q + e^{\mu}[(D-1)\mu_{;c}{}^c - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}] \tag{A.40}$$

where in the RHS of (A.39), (A.40) index raising operations and covariant derivatives are all computed using the metric q.

Let us now see how we can use the relations derived above to obtain (**??**): the dimensional reduction of the Einstein action from $(D+1)$ dimensions to D dimensions, where we wish to end up with the Einstein action in D dimensions plus a scalar.

The Einstein action in $D+1$ dimensions is

$$S = \frac{1}{16\pi G}\int d^{D+1}x\sqrt{-g_{D+1}}R_{D+1} \tag{A.41}$$

We wish to compactify the direction x^{D+1} on a circle of coordinate length B:

$$0 \le x^{D+1} < B \tag{A.42}$$

We write

$$g_{D+1,D+1} = e^C \tag{A.43}$$

Using (A.31), we find

$$S_{D+1} = \frac{B}{16\pi G}\int d^Dx\sqrt{-g_D}e^{\frac{1}{2}C}[R]_D \tag{A.44}$$

We now wish to absorb the factor $e^{\frac{1}{2}C}$ into the curvature term, through a definition

$$g^E_{ab} = e^{\mu C}g_{ab} \tag{A.45}$$

To see what value of μ we should take, we count the powers of the metric in (A.55):

(i) The determinant of the metric g_D has D powers of the metric g_{ab}. Thus the factor $\sqrt{-g_D}$ has $\frac{D}{2}$ powers of the metric.

(ii) The connection Γ^a_{bc} has no net powers of the metric: in the expression (A.5) there is one power of the *inverse* metric $g^{\lambda\kappa}$ and one power of the metric througfh the terms like $g_{\kappa\nu,\lambda}$. The Riemann tensor (A.6) then has no net powers of the metric either. The Ricci tensor (A.7) is obtained by contracting an up and a down index in the Riemann tensor, and so still has no net powers of theg metric. Finally, The Ricci scalar (A.8) has one power of the inverse metric more than the Ricci tensor; thus it has -1 powers of the metric g_{ab}

(iii) Thus we have a total of

$$\frac{D}{2} - 1 = \frac{D-2}{2} \tag{A.46}$$

powers of the metric g_{ab} in the action (A.55). we need to absorb a factor $e^{\frac{1}{2}C}$. Thus we write

$$(g^E_{ab})^{\frac{D-2}{2}} = e^{\frac{1}{2}C}(g_{ab})^{\frac{D-2}{2}} \tag{A.47}$$

which gives

$$g^E_{ab} = e^{\mu} g_{ab} \tag{A.48}$$

with

$$\mu = \frac{C}{D-2} \tag{A.49}$$

We can now use the relation (A.40), where $q_{ab} = g^E_{ab}$, and μ is given by (A.49). We find

$$[R]_D = e^{\frac{2}{D-2}C}\left([R^E]_D + (D-1)\mu_{;c}{}^{c} - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}\right) \tag{A.50}$$

Thus the action (**??**) is

$$\begin{aligned} S_{D+1} &= \frac{B}{16\pi G}\int d^D x\sqrt{-g_D}e^{\frac{1}{2}C}[R]_D \\ &= \frac{B}{16\pi G}\int d^D x\sqrt{-g^E_D}\left([R^E]_D + (D-1)\mu_{;c}{}^{c} - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^D x\sqrt{-g^E_D}\left([R^E]_D - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^D x\sqrt{-g^E_D}\left([R^E]_D - \frac{(D-1)}{4(D-2)}C_{,c}C^{,c}\right) \end{aligned} \tag{A.51}$$

Here in the second step we observe the cancellation of powers of e^C (which was arranged to happen by the choice (A.49)) and in the third step we have noted that $\mu_{;c}{}^{c}$ is a total divergence and can hence be dropped from the action.

We now see that the action S_{D+1} has the symmetry (**??**)

$$\begin{aligned} C &\to -C \\ g^E_{\mu\nu} &\to g^E_{\mu\nu} \end{aligned} \tag{A.52}$$

A.4 Two compact directions

We obtained the symmetry (A.52) by compactifying one directions and obtaining the scalar C. We will now compactify an additional direction, so that we will have two scalars $C, \tilde{C}$. We will then look for a symmetry that generalizes the map $C \to -C$ to a more general linear map mixing $C, \tilde{C}$. We proceed in the following steps:

(i) We again start with the D+1 dimensional action

$$S_{D+1} = \int d^{D+1}x\sqrt{-g}R \tag{A.53}$$

We compactify the direction x^{D+1} as in (A.42), and write

$$g_{D+1,D+1} = e^{C} \tag{A.54}$$

At this stage we get the action (A.55)

$$S_{D+1} = \frac{B}{16\pi G}\int d^D x\sqrt{-g_D}e^{\frac{1}{2}C}[R]_D \tag{A.55}$$

(ii) We wish to define a rescaled metric for the D dimensional theory. In its appropriate setting in string theory, this metric will be called the 'string metric', so we denote it g^S_{ab}. We write

$$g^S = e^{\alpha C} g \tag{A.56}$$

In the relation (A.32), we see that $q_{ab} = g^S_{ab}$, and

$$\mu = \alpha C \tag{A.57}$$

We see that

$$\sqrt{-g_D} = e^{-\frac{D}{2}\alpha C}\sqrt{-g^S_D} \tag{A.58}$$

Using (A.40), we find

$$S_{D+1} = \int d^D x\sqrt{-g_N}e^{-\frac{D}{2}\alpha C}e^{\frac{C}{2}}e^{\alpha C}\left(R^S_D + (D-1)\alpha\partial^2 C - \frac{(D-1)(D-2)}{4}\alpha^2\partial C\partial C\right) \tag{A.59}$$

The term with $\partial^2 C$ can be integrated by parts to give

$$S_{D+1} = \int d^D x\sqrt{-g^S_D}e^{[-\frac{(D-2)}{2}\alpha+\frac{1}{2}]C}\left(R^S_D - (\frac{(D-1)}{2}\alpha - \frac{(D-1)(D-2)}{4}\alpha^2)\partial C\partial C\right) \tag{A.60}$$

(iii) We now wish to compactify an additional direction x^{D-1}

$$0 \le x^{D-1} < A \tag{A.61}$$

We set

$$g^S_{(D-1)(D-1)} = e^{\tilde{C}} \tag{A.62}$$

and assume as before that

$$g^S_{(D-1),a} = 0, \quad a = 0, 1, \dots (D-2) \tag{A.63}$$

Now we have a $D-1$ dimensional spacetime, carrying a metric g^S_{ab} with $a, b = 0, 1, \dots (D-2)$, and two scalars $C, \tilde{C}$.

$$\sqrt{-g^S_D} = e^{\frac{\tilde{C}}{2}}\sqrt{-g^S_{(D-1)}} \tag{A.64}$$

Using (A.31) to write R^S_D to R^S_{D-1}, we find that the action reduced to $(D-1)$ dimensions is

$$\begin{aligned} S_{D+1} &= \int d^D x\sqrt{-g^S_D} e^{[-\frac{(D-2)}{2}\alpha+\frac{1}{2}]C}(R^S_D \\ &\quad - \left[\frac{(D-1)}{2}\alpha - \frac{(D-1)(D-2)}{4}\alpha^2\right]\partial C\partial C) \\ &= \int d^{(D-1)}x\sqrt{-g^S_{(D-1)}}e^{\frac{\tilde{C}}{2}}e^{[-\frac{(D-2)}{2}\alpha+\frac{1}{2}]C}(R^S_{(D-1)} - \partial^2\tilde{C} - \frac{1}{2}\partial\tilde{C}\partial\tilde{C} \\ &\quad -(\frac{(D-1)}{2}\alpha - \frac{(D-1)(D-2)}{4}\alpha^2)\partial C\partial C) \end{aligned} \tag{A.65}$$

We integrate by parts the term containing $\partial^2\tilde{C}$, to find

$$\begin{aligned} S_{D+1} &= \int d^{(D-1)}x\sqrt{-g^S_{(D-1)}}e^{\frac{\tilde{C}}{2}}e^{[-\frac{(D-2)}{2}\alpha+\frac{1}{2}]C}[R^S_{(D-1)} \\ &+ [-\frac{(D-2)}{2}\alpha+\frac{1}{2}]\partial C\partial\tilde{C} - [\frac{(D-1)}{2}\alpha - \frac{(D-1)(D-2)}{4}\alpha^2]\partial C\partial C] \end{aligned} \tag{A.66}$$

(iv) We now look for a symmetry of the form

$$\begin{aligned} \tilde{C}' &= (a\tilde{C}+bC) \\ C' &= (c\tilde{C}+dC) \\ g'_{N,(D-1)} &= g_{N,(D-1)} \end{aligned} \tag{A.67}$$

and ask that

$$S'_{D+1} = S_{D+1} \tag{A.68}$$

We have 5 free parameters: α, a, b, c, d. Requiring that the exponential prefactor in (A.66) remain unchanged under the map (A.67) gives

$$\frac{1}{2}(a\tilde{C}+bC) + [-\frac{(D-2)}{2}\alpha+\frac{1}{2}](c\tilde{C}+dC) = \frac{1}{2}\tilde{C} + [-\frac{(D-2)}{2}\alpha+\frac{1}{2}]C \tag{A.69}$$

which yields two constraints

$$\frac{1}{2}a + [-\frac{(D-2)}{2}\alpha+\frac{1}{2}]c = \frac{1}{2} \tag{A.70}$$

$$\frac{1}{2}b + [-\frac{(D-2)}{2}\alpha+\frac{1}{2}]d = [-\frac{(D-2)}{2}\alpha+\frac{1}{2}] \tag{A.71}$$

The curvature scalar R^S_{D-1} remains unchanged under the map (A.67). Requiring that the coefficients of the kinetic terms $\partial C\partial C$, $\partial C\partial\tilde{C}$ and $\partial\tilde{C}\partial\tilde{C}$ remain unchanged gives the three relations:

$$\left(-\frac{(D-2)}{2}\alpha+\frac{1}{2}\right)bd \quad - \quad \left(\frac{(D-1)}{2}\alpha-\frac{(D-1)(D-2)}{4}\alpha^2\right)d^2$$
$$= \quad -\left(\frac{(D-1)}{2}\alpha-\frac{(D-1)(D-2)}{4}\alpha^2\right) \tag{A.72}$$

$$\left(-\frac{(D-2)}{2}\alpha+\frac{1}{2}\right)(ad+bc) \quad - \quad \left(\frac{(D-1)}{2}\alpha-\frac{(D-1)(D-2)}{4}\alpha^2\right)(2cd)$$
$$= \quad \left(-\frac{(D-2)}{2}\alpha+\frac{1}{2}\right) \tag{A.73}$$

$$\left(-\frac{(D-2)}{2}\alpha+\frac{1}{2}\right)ac-\left(\frac{(D-1)}{2}\alpha-\frac{(D-1)(D-2)}{4}\alpha^2\right)c^2=0 \tag{A.74}$$

(v) We thus have 5 relations (A.70)-(A.74) for the 5 parameters α, a, b, c, d. These relations must of course admit the trivial solution where nothing is changed

$$\alpha=0, \quad a=1, \quad b=0, \quad c=0 \quad d=1 \tag{A.75}$$

What is interesting is that are also nontrivial solutions

$$\alpha=\frac{1\pm\sqrt{D-1}}{D-2}, \quad a=-1, \quad b=0, \quad c=\mp\frac{2}{\sqrt{D-1}}, \quad d=1 \tag{A.76}$$

This is the T-duality symmetry noted in section **??**.

A.4.1 The IIA and IIB string actions

We choose the positive sign in (A.76); the other choice gives a combination of T-duality and other symmetries. We set $D=10$. This gives

$$\alpha=\frac{1}{2} \tag{A.77}$$

Recalling the definition of the dilaton

$$C=\frac{4}{3}\phi \tag{A.78}$$

we find that the string metric g^S_{ab} is related to 11-d metric g_{ab} through

$$g^S_{ab}=e^{\frac{C}{2}}g_{ab} \tag{A.79}$$

The 10-d action in terms of the metric g_{ab} is given by eq.(A.55))

$$S=\frac{B}{16\pi G}\int d^{10}x\sqrt{-g}e^{\frac{1}{2}C}[R]_{10} \tag{A.80}$$

We use the substitution (A.79), in (??), so that

$$\mu = \frac{1}{2}C \tag{A.81}$$

This gives

$$[R]_{10} = e^{\frac{C}{2}}\left([R]_S + \frac{9}{2}C_{;c}{}^{c} - \frac{9}{2}C_{,c}C^{,c}\right) \tag{A.82}$$

Thus

$$\begin{aligned} S &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^S}e^{-\frac{5}{2}C}e^{\frac{1}{2}C}e^{\frac{1}{2}}\left([R]_S + \frac{9}{2}C_{;c}{}^{c} - \frac{9}{2}C_{,c}C^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^S}e^{-\frac{3}{2}C}\left([R]_S + \frac{9}{2}C_{;c}{}^{c} - \frac{9}{2}C_{,c}C^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^S}e^{-\frac{3}{2}C}\left([R]_S + \frac{9}{4}C_{,c}C^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^S}e^{-2\phi}\left([R]_S + 4\phi_{,c}\phi^{,c}\right) \end{aligned} \tag{A.83}$$

This is the action of IIA string theory, which is obtained from the 11-d M theory by compactification of x^{10}. But we can see that as far as the metric g^S_{ab} and dilaton ϕ are concerned, this is also the action for the IIB theory. This is seen as follows:

(i) Compactify one further direction x^9, obtaining the action (A.66) (with $D = 10$ and $\alpha = 1/2$). This is still the IIA theory.

(ii) Perform a T-duality in the direction x^9. This brings us to the IIB theory. But the action (A.66) remains invariant under this duality, the duality map (??) was constructed to get this invariance.

(iii) We can now regard the action (A.66) again as an action for a 10-d theory, getting (A.83). Thus (??) is also the action fo 10-d IIB theory.

Let us convert the 'string frame' action (A.83) to the Einstein frame. We have

$$e^{-2\phi}\left(g^S_{ab}\right)^4 = \left(g^E_{ab}\right)^4 \tag{A.84}$$

so that

$$g^E_{ab} = e^{-\frac{\phi}{2}}g^S_{ab} \tag{A.85}$$

Thus we have a scaling with $\mu = -\frac{1}{2}\phi$. We get

$$[R]_S = e^{-\frac{1}{2}\phi}\left([R]_E - \frac{9}{2}\phi_{;c}{}^{c} - \frac{9}{2}\phi_{,c}\phi^{,c}\right) \tag{A.86}$$

This gives

$$\begin{aligned} S &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^S}e^{-2\phi}\left([R]_S + 4\phi_{,c}\phi^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^E}e^{\frac{5}{2}\phi}e^{-2\phi}e^{-\frac{1}{2}\phi}\left([R]_E - \frac{9}{2}\phi_{;c}{}^{c} - \frac{9}{2}\phi_{,c}\phi^{,c} + 4\phi_{,c}\phi^{,c}\right) \\ &= \frac{B}{16\pi G}\int d^{10}x\sqrt{-g^E}\left([R]_E - \frac{1}{2}\phi_{,c}\phi^{,c}\right) \end{aligned} \tag{A.87}$$

A.5 The Casimir effect

Consider a free, massless scalar field ϕ in 1+1 dimensional spacetime x, t. Let x be compactified to a circle

$$0 \le x < L \tag{A.88}$$

We have seen that the scalar field ϕ can be decompsed into a set of independent harmonic oscillators, with frequencies

$$\omega_n = \frac{2\pi|n|}{L}, \quad -\infty < n < \infty \tag{A.89}$$

Each harmonic oscillator has a ground state energy $\frac{1}{2}\omega_n$, so that the ground state energy of the entire system is

$$E_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\omega_n \tag{A.90}$$

This sum clearly diverges. To deal with this issue, we assume that some physical effect cuts off the sum at very high energies. Setting this cutoff scale as E_c, we assume that the sum (A.90) should be replaced by a regularized sum

$$E_0^R = \frac{1}{2}\sum_{-\infty}^{\infty}\omega_n f(\omega_n) \tag{A.91}$$

Here the function $f(E)$ has the properties

$$\begin{aligned} f(E) &\to 1, \quad E \ll E_c \\ f(E) &\to 0, \quad E \gg E_c \end{aligned} \tag{A.92}$$

We also assume that f is smooth; i.e., it drops very gently from its value 1 at $E \ll E_c$ to its value 0 at $E \gg E_c$. We will take the limit $E_c \to \infty$ at the end, and so this smoothness can be stated as

$$\begin{aligned} \left|\frac{d^n f(E)}{dE^n}\right| &\ll \frac{C_n}{E_c^n}, \quad E \ll E_c \\ \left|\frac{d^n f(E)}{dE^n}\right| &\ll \frac{D_n}{E^n}, \quad E \gg E_c \end{aligned} \tag{A.93}$$

for constants C_k, D_k of order unity. An example for the function f satisfying these conditions is

$$f(E) = e^{-\frac{E}{E_c}} \tag{A.94}$$

We find that

$$E_0^R = \frac{1}{2}\sum_{-\infty}^{\infty}\frac{2\pi|n|}{L}f(\omega_n) = \frac{2\pi}{L}\sum_{n=1}^{\infty}nf(\frac{2\pi n}{L}) \tag{A.95}$$

The energy (A.91) will still diverge when we take the limit $E_c \to \infty$. But we can require that there be a basic negative energy shift to the Hamiltonian, which will cancel the divergent part of this energy. In the limit where the compactification circle has an infinite length, we are in 1+1 dimensional Minkowski space, and we assume that the negative energy shift is such that the energy of the vacuum in this case is zero. To reach this case, assume that the spatial circle x is compactified to a very large length; we will call this length L_∞ since we will take L_∞ at the end. Let $E_0^{R,\infty}$ be the analogue of (A.91) on this circle of length L

$$E_0^{R,\infty} = \frac{2\pi}{L_\infty}\sum_{n=1}^{\infty}nf(\frac{2\pi n}{L_\infty}) \tag{A.96}$$

Then the ground state energy per unit length will be E_∞^R/L_∞, and this is the value we must subtract per unit length of our system. Thus the net ground state energy for our compactification (A.88) will be

$$\Delta E_0 = E_0^R - \frac{L}{L_\infty}E_0^{R,\infty} \tag{A.97}$$

This is the quantity that we will compute now, keeping in mind that we must set $E_c \to \infty$ and $L_\infty \to \infty$ in the defuinition of ΔE_0.

A.5.1 Writing $E_R^{R,\infty}$ as an integral

For the compactification (A.88), the separation between energy levels is $2\pi/L$. For the compactification length L_∞, the separation $2\pi/L_\infty$ goes to zero in the limit $L_\infty \to \infty$, so we can replace the sum over n by an *integral*. The net energy (A.97) will then arise as the difference between a sum and an integral.

We write

$$f(\frac{2\pi n}{L}) = \tilde{f}(n) \tag{A.98}$$

so that (A.95) becomes

$$E_0^R = \frac{2\pi}{L}\sum_{n=1}^{\infty}n\tilde{f}(n) \tag{A.99}$$

In (A.96) we have the regularization function

$$f(\frac{2\pi n}{L_\infty}) = f(\frac{2\pi n}{L}\frac{L}{L_\infty}) \equiv \tilde{f}(\frac{L}{L_\infty}n) \tag{A.100}$$

We write

$$\frac{L}{L_\infty} n \equiv x \tag{A.101}$$

The sum in (A.96) goes over to an integral

$$\sum_n \to \int dn = \frac{L_\infty}{L} dx \tag{A.102}$$

We get

$$E_0^{R,\infty} = \left(\frac{2\pi}{L_\infty}\right)\left(\frac{L_\infty}{L}\right)^2 \int_{x=0}^{\infty} dx x\tilde{f}(x) = \frac{2\pi L_\infty}{L^2}\int_{x=0}^{\infty} dx x\tilde{f}(x) \tag{A.103}$$

Thus

$$\Delta E_0 = E_0^R - \frac{L}{L_\infty} E_0^{R,\infty} = \frac{2\pi}{L}\left(\sum_{n=1}^{\infty} n\tilde{f}(n) - \int_{x=0}^{\infty} dx x\tilde{f}(x)\right) \equiv \frac{2\pi}{L}(S - I) \tag{A.104}$$

where we have called the sum S and the integral I.

A.5.2 Computing $S - I$

The quantity $S - I$ is depicted pictorially in fig.**??**. We plot the function $x\tilde{f}(x)$ vs x. Since $\tilde{f}(x) \to 1$ for small x, the function $x\tilde{f}(x)$ rises linearly in this region. But at $x \gg 1$, the function $x\tilde{f}(x)$ damps smoothly to zero because $\tilde{f}(x) \to 0$.

Consider the region

$$n \le x < n + 1 \tag{A.105}$$

The sum S has a term $S_n = n\tilde{f}(n)$, which we can represent by the area of a rectangle of height $n\tilde{f}(n)$ over this region. The integral I on the other hand has a contribution I_n given by the area under the curve $x\tilde{f}(x)$ in this region. Thus the contribution to $S_n - I_n$ from this region is given by the area of the shaded area which lies between the curve and the rectangle.

From fig.**??** we see that $S_n - I_n$ is negative for small n, and it may seem that the sum over n is diverging to negative infinity. But $S_n - I_n$ is positive at large n where the function $x\tilde{f}(x)$ is decreasing. Remarkably, the sum over n leads to a definite finite value for $\sum_n (S_n - I_n)$, with this value being independent of the precise choice of function $\tilde{f}$. This independence of regularization is of course an aspect of the renormalizability of the theory, which allows the existence of well-defined low energy physics after a subtraction of infinities arising from high energy cutoffs.

To compute I_n, we expand the function $x\tilde{f}(x)$ around $x = n$:

$$\begin{aligned} I_n &= \int_n^{n+1} x\tilde{f}(x) dx \\ &= \int_0^1 dy \left([n\tilde{f}(n)] + [n\tilde{f}'(n) + \tilde{f}(n)]y + [2\tilde{f}'(n) + n\tilde{f}''(n)]\frac{y^2}{2}] + \ldots \right) \\ &= [nf(n)] + \frac{1}{2}[n\tilde{\tilde{f}}'(n) + \tilde{f}(n)] + \frac{1}{6}[2\tilde{f}'(n) + n\tilde{f}''(n)] + \ldots \end{aligned} \tag{A.106}$$

Thus

$$S - I = -\sum_{n=0}^{\infty}\left(\frac{1}{2}[n\tilde{f}'(n) + \tilde{f}(n)] - \frac{1}{6}[2\tilde{f}'(n) + n\tilde{f}''(n)]\right) \tag{A.107}$$

We write

$$\tilde{S} = \sum_{n=0}^{\infty}[n\tilde{f}'(n) + \tilde{f}(n)] \tag{A.108}$$

In general a sum is hard to compute, but an integral can be easier. Thus let us define

$$\tilde{I} = \int_0^{\infty} dx[x\tilde{f}'(x) + \tilde{f}(x)] \tag{A.109}$$

This integral happens to vanish

$$\tilde{I} = \int_0^{\infty} dx[x\tilde{f}(x)]' = [x\tilde{f}(x)]_0^{\infty} = 0 \tag{A.110}$$

where we have used the vanishing of $\tilde{f}$ at $x \to \infty$.

We have

$$\tilde{S} = \tilde{I} + (\tilde{S} - \tilde{I}) = (\tilde{S} - \tilde{I}) \tag{A.111}$$

where the quantity $(\tilde{S}-\tilde{I})$ is again the difference between a sum and an integral. Let us compute this difference in the same way as above, expanding the function $[x\tilde{f}'(x) + \tilde{f}(x)]$ in a series about $x = n$, With $y = x - n$:

$$[x\tilde{f}'(x) + \tilde{f}(x)] = [n\tilde{f}'(n) + \tilde{f}(n)] + [2\tilde{f}'(n) + n\tilde{f}''(n)]y + \dots \tag{A.112}$$

We have

$$\begin{aligned}(\tilde{S} - \tilde{I}) &= \sum_{n=0}^{\infty}([n\tilde{f}'(n) + \tilde{f}(n)] \\ &\quad - \int_{y=0}^{1} dy\left([n\tilde{f}'(n) + \tilde{f}(n)] + [2\tilde{f}'(n) + nf''(n)]y + \dots\right)) \\ &= -\sum_{n=0}^{\infty}\frac{1}{2}[2\tilde{f}'(n) + n\tilde{f}''(n)]\end{aligned} \tag{A.113}$$

Using this expression in (A.111), and then substituting for $\tilde{S}$ in (A.107), we get

$$\begin{aligned}S - I &= \sum_{n=0}^{\infty}\left(\frac{1}{4}[2\tilde{f}'(n) + n\tilde{f}''(n)] + \frac{1}{6}[2\tilde{f}'(n) + n\tilde{f}''(n)]\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{6}\tilde{f}'(n) + \frac{1}{12}n\tilde{f}''(n)\right)\end{aligned} \tag{A.114}$$

If we approximate this sum as an integral, we get

$$\begin{aligned}
S - I \quad &\rightarrow \quad \int_0^\infty dx[\frac{1}{6}\tilde{f}'(x) + \frac{1}{12}x\tilde{f}''(x)] \\
&= \quad \frac{1}{6}\tilde{f}|_0^\infty + \frac{1}{12}[x\tilde{f}']_0^\infty - \int_0^\infty dx\tilde{f}'(x) \\
&= \quad -\frac{1}{6} + \frac{1}{12} \\
&= \quad -\frac{1}{12} \qquad \text{(A.115)}
\end{aligned}$$

where we have used that $\tilde{f}(0) = 1, \tilde{f}'(0) = 0$.

One may think that we again have to worry about the difference between the sum (A.114) and the integral (A.115), but this is not the case; this time the difference can be ignored. Consider the difference Δ_n in the segment $n \leq x < n+1$:

$$\begin{aligned}
\Delta_n \quad &= \quad [\frac{1}{6}\tilde{f}'(n) + \frac{1}{12}n\tilde{f}''(n)] \\
&\quad - \int_{y=0}^1 dy([\frac{1}{6}\tilde{f}'(n) + \frac{1}{12}n\tilde{f}''(n)] + [\frac{1}{4}\tilde{f}''(n) + \frac{1}{12}\tilde{f}'''(n)]y + \ldots) \\
&= \quad \frac{1}{2}[\frac{1}{4}\tilde{f}''(n) + \frac{1}{12}n\tilde{f}'''(n)] + \ldots \qquad \text{(A.116)}
\end{aligned}$$

From (A.93) and the definition (A.104) of x, we find

$$\begin{aligned}
\frac{d\tilde{f}(x)}{dx} \quad &= \quad \frac{d}{dx}f(\frac{2\pi}{l}x) = \frac{2\pi}{L}f'(\frac{2\pi}{L}x) \\
&< \quad \frac{2\pi}{L}\frac{1}{E_c}, \qquad x \gtrsim \frac{LE_c}{2\pi} \\
&< \quad \frac{1}{x} \qquad x \lesssim \frac{LE_c}{2\pi} \qquad \text{(A.117)}
\end{aligned}$$

where f' denotes the derivative of f with respect to its argument. Similarly, we find

$$\begin{aligned}
\frac{d^2\tilde{f}(x)}{dx^2} \quad &< \quad \left(\frac{2\pi}{LE_c}\right)^2, \qquad x \gtrsim \frac{LE_c}{2\pi} \\
&< \quad \frac{1}{x^2} \qquad x \lesssim \frac{LE_c}{2\pi} \qquad \text{(A.118)}
\end{aligned}$$

$$\begin{aligned}
\frac{d^3\tilde{f}(x)}{dx^3} \quad &< \quad \left(\frac{2\pi}{LE_c}\right)^3, \qquad x \gtrsim \frac{LE_c}{2\pi} \\
&< \quad \frac{1}{x^3} \qquad x \lesssim \frac{LE_c}{2\pi} \qquad \text{(A.119)}
\end{aligned}$$

and so on. We then find that in the limit $E_c \rightarrow \infty$, the sum $\sum_n \Delta_n$ goes to zero. The reason for this vanishing can be traced to the fact that the cutoff function

f is very smooth (varying on the scale E_c), so once if we have an expression with sufficiently many derivatives of f, then it vanishes when the energy cutoff scale is taken to infinity.

A.5.3 Collecting the results

Our goal was to compute the ground state energy ΔE_0 for a scalar field on a circle of length L, where the Hamiltonian was defined in such a way that the ground state energy in the limit $L \to \infty$ was zero. From (A.104) and (A.115) we find

$$\begin{aligned} \Delta E_0 &= \frac{2\pi}{L}(S - I) \\ &= -\left(\frac{2\pi}{L}\right)\left(\frac{1}{12}\right) \end{aligned} \qquad \text{(A.120)}$$

To summarize, the ground state energy for a scalar field involves a sum (A.90) which, apart from overall factors, involves the divergent sum

$$\mathcal{B} = 1 + 2 + \ldots \qquad \text{(A.121)}$$

After regularizing this sum as in (A.91), and renormalizing it by a subtraction as in (A.97), we get

$$\mathcal{B} = 1 + 2 + \ldots \to -\frac{1}{12} \qquad \text{(A.122)}$$

Once we know that there is a well defined renormalized value for $\mathcal{B}$, we can obtain it by simpler mathematical methods. The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{(A.123)}$$

so that

$$\mathcal{B} = \zeta(-1) = -\frac{1}{12} \qquad \text{(A.124)}$$

where an analytical continuation is required from values $s > 1$.

A.5.4 Ground state energy for fermions

Let us now consider the ground state energy ΔE_0 for the case where the we a fermionic field ψ instead of the bosonic field ϕ on or circle of length L.

The fermion field can also be decomposed into harmonic oscillators, and we get a similar contribution to the ΔE_0 from the ground state energy of each oscillator.

For a usual harmonic oscillator defined in terms of a 'bosonic' variable x, we have the Hamiltonian

$$H_{boson} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) = \frac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\omega = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega \qquad \text{(A.125)}$$

where we have used the commutator $[\hat{a}, \hat{a}^\dagger] = 1$. This gives the ground state energy

$$E_0^{bosonic} = \frac{1}{2}\omega \tag{A.126}$$

which we used in the derivation of (A.120).

We can also define an oscillator for a fermionic variable, which gives

$$H_{fermion} = \frac{1}{2}(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b})\omega = (\hat{b}^\dagger\hat{b} - \frac{1}{2})\omega \tag{A.127}$$

where we have used the anti-commutator $[\hat{b}, \hat{b}^\dagger] = -1$. This gives a ground state energy

$$E_0^{fermionic} = -\frac{1}{2}\omega \tag{A.128}$$

For the computation of ΔE_0 for the field ψ, we now have two cases:

(i) **Ramond boundary condition**: The field ψ is periodic around the circle

$$\psi(x + L) = \psi(x) \tag{A.129}$$

We get the same mode frequencies as in the bosonic case

$$\omega_n = \frac{2\pi|n|}{L}, \quad -\infty < n < \infty \tag{A.130}$$

The computation of ΔE_0 is then the same as the bosonic case, but due to the negative sign in (A.128) we get in place of (A.120)

$$\Delta E_0 = -\left(\frac{2\pi}{L}\right)(1 + 2 + \dots) = \left(\frac{2\pi}{L}\right)\left(\frac{1}{12}\right) \tag{A.131}$$

(ii) **Neveu-Schwarz boundary condition**: The field ψ is anti-periodic around the circle

$$\psi(x + L) = -\psi(x) \tag{A.132}$$

We get the mode frequencies

$$\omega_n = \frac{2\pi|n + \frac{1}{2}||}{L}, \quad -\infty < n < \infty \tag{A.133}$$

Factoring out $\frac{2\pi}{L}$ as before, we find that in place of the sum $1 + 2 + \dots$ we get the sum

$$\mathcal{F} = \frac{1}{2} + \frac{3}{2} + \dots \tag{A.134}$$

We can compute $\mathcal{F}$ by regularizing and renormalizing as before, but we can obtain it more quickly by relating it to $\mathcal{B}$. We assume that the sum has been regularized, though we do not write the cutoff function f in what follows. We find

$$\begin{aligned} 2\mathcal{F} &= 1+3+5+\dots \\ 2\mathcal{B} &= 2+4+6+\dots \end{aligned} \tag{A.135}$$

Thus

$$2\mathcal{F}+2\mathcal{B} = 1+2+3+\cdots = \mathcal{B} \tag{A.136}$$

which gives

$$\mathcal{F} = -\frac{1}{2}\mathcal{B} = \frac{1}{24} \tag{A.137}$$

Noting the negative sign in (A.128) fir the ground state energy of an oscillator, we find that for an antiperiodic fermion ψ

$$\Delta E_0 = -\left(\frac{2\pi}{L}\right)\left(\frac{1}{2}+\frac{3}{2}+\dots\right) = -\left(\frac{2\pi}{L}\right)\left(\frac{1}{24}\right) \tag{A.138}$$

Appendix B

The idea of particle creation in curved space

B.1 The Schwarzschild hole

Let us start with the Schwarzschild metric of the 3+1 dimensional black hole

$$ds^2 = -(1 - \frac{2GM}{c^2 r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{B.1}$$

We will set

$$G = c = \hbar = 1 \tag{B.2}$$

and write $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ for the metric on the unit 2-sphere S^2. Then (B.1) becomes

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \tag{B.3}$$

Consider the line

$$r = r_0, \quad \theta = \theta_0, \quad \phi = \phi_0 \tag{B.4}$$

so that only t changes along this line

(i) For $r > 2M$ the metric along this line gives

$$ds^2 = -(1 - \frac{2M}{r})dt^2 < 0 \tag{B.5}$$

so this is a timelike line, and can be the worldline of an actual particle.

(ii) For $r < 2M$ we get

$$ds^2 = -(1 - \frac{2M}{r})dt^2 > 0 \tag{B.6}$$

so this is a spacelike line, and *cannot* be the path of a particle. In other words, a particle cannot sit at constant r, θ, ϕ for $r < 2M$.

The surface $r = 2M$ is called the *horizon*. Classically (i.e. without quantum effects) no particle can emerge from inside the horizon to the outside.

B.2 Kruskal coordinates

In the metric (B.3) we see that there is a problem when $1 - \frac{2M}{r} = 0$, since the coefficient of dt^2 vanishes and the coefficient of dr^2 diverges. Before we know any more, we cannot be sure if this means that the coordinates are bad at this location $r = 2M$ or if the metric has a geometrical singularity of some kind. It will turn out that the singularity at the horizon is only a coordinate singularity. To show this, we need to use coordinates that are well behaved at the horizon. Let us find such coordinates.

(i) First we look at the t, r part of the metric and write

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} = (1 - \frac{2M}{r})[-dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})^2}] \tag{B.7}$$

We would now like to find a coordinate r^* such that

$$dr^{*2} = \frac{dr^2}{(1 - \frac{2M}{r})^2} \tag{B.8}$$

This gives the equation $dr^* = \frac{dr}{1-\frac{2M}{r}}$ which has the solution

$$r^* = \int^r \frac{dr}{1 - \frac{2M}{r}} = \int^r \frac{r}{r - 2M} = \int^r dr[1 + \frac{1}{\frac{r}{2M} - 1}] = r + 2M\ln(\frac{r}{2M} - 1) \tag{B.9}$$

where we have set the arbitrary additive constant to zero. The metric (B.3) becomes

$$ds^2 = (1 - \frac{2M}{r})[-dt^2 + dr^{*2}] + r^2 d\Omega_2^2 \tag{B.10}$$

(ii) Now we move to null coordinates by writing

$$u = t + r^*, \quad v = t - r^* \tag{B.11}$$

This gives

$$ds^2 = (1 - \frac{2M}{r})[-du dv] + r^2 d\Omega_2^2 \tag{B.12}$$

(iii) Let us now look at the ranges of these coordinates. Note that the range $r = (2M, \infty)$ maps to $r^* = (-\infty, \infty)$. Now consider a null geodesic falling radially into the hole. Thus θ, ϕ are constant, and the worldline will be given by solving $ds^2 = 0$ in the (t, r^*) space. At infinity where the metric is flat the ingoing geodesic is $t + r = const.$. From (B.12) we see that taking into account the metric of the hole changes this to

$$t + r^* = u = u_0 \tag{B.13}$$

By taking geodesics starting from a given r^* with different values of t we see that we can cover the full range $-\infty < u_0 < \infty$ for points outside the horizon.

Similarly, $v = t - r^*$ can cover this full range. But note in addition that as the ingoing null geodesic approaches the horizon we get

$$v = t - r^* = u_0 - 2r^* \to \infty \tag{B.14}$$

In short, the 'future horizon' (i.e. the horizon which is crossed in the future by an observer who decides to fall into the black hole) is given by

$$-\infty < u < \infty, \quad v = \infty \tag{B.15}$$

(iv) From (B.15) we see that our coordinates (u, v) 'end' at the horizon. If we wish to see the horizon as a regular region of our manifold, then we would like to have coordinates that smoothly take us across the horizon. Thus we need the horizon to be at *finite* values of our coordinates, unlike (B.15). Let we write

$$U = e^{\alpha u}, \quad V = -e^{-\alpha v} \tag{B.16}$$

where we will choose the constant α later. Assuming $\alpha > 0$, we see that the region outside the horizon is

$$U > 0, \quad V < 0 \tag{B.17}$$

and the horizon itself is

$$0 < U < \infty, \quad V = 0 \tag{B.18}$$

Thus we have brought the horizon to a finite position in our new coordinates U, V, and if the metric is smooth at $U = V = 0$ then we can continue the spacetime past the region (B.17).

(v) From (B.16) we get

$$dU = \alpha e^{\alpha u} du, \quad dV = \alpha e^{-\alpha v} dv \tag{B.19}$$

Thus the metric (B.12) becomes

$$ds^2 = -(1 - \frac{2M}{r}) \frac{e^{-\alpha(u-v)}}{\alpha^2} dU dV + r^2 d\Omega_2^2 = -\frac{(r-2M)}{r} \frac{e^{-\alpha(u-v)}}{\alpha^2} dU dV + r^2 d\Omega_2^2 \tag{B.20}$$

Now note that

$$e^{-\alpha(u-v)} = e^{-2\alpha r^*} = e^{-2\alpha[r + 2M \ln(\frac{r}{2M} - 1)]} = e^{-2\alpha r} (\frac{r}{2M} - 1)^{-4\alpha M} = e^{-2\alpha r} (2M)^{4\alpha M} (r - 2M)^{-4\alpha M} \tag{B.21}$$

We now see that if we choose

$$\alpha = \frac{1}{4M} \tag{B.22}$$

then we cancel the factor $r - 2M$ in (B.20), getting

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega^2 \tag{B.23}$$

This metric is now written in coordinates U, V, θ, ϕ, with

$$\begin{aligned} U &= (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{\frac{t}{4M}} \\ V &= -(\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{-\frac{t}{4M}} \end{aligned} \tag{B.24}$$

Note that

$$UV = -(\frac{r}{2M} - 1) e^{\frac{r}{2M}} \tag{B.25}$$

and we should understand the symbol r in (B.23) as the function $r(U, V)$ given through the transcendental equation (B.25). Since we do not need the explicit form of this function for our analysis, we leave it as the symbol r. All we note for now is that at the horizon (where we are trying to get smooth coordinates) the function r is a smooth function on the manifold, with $r \approx 2M(1 - UV)$.

B.3 Extending past the horizon

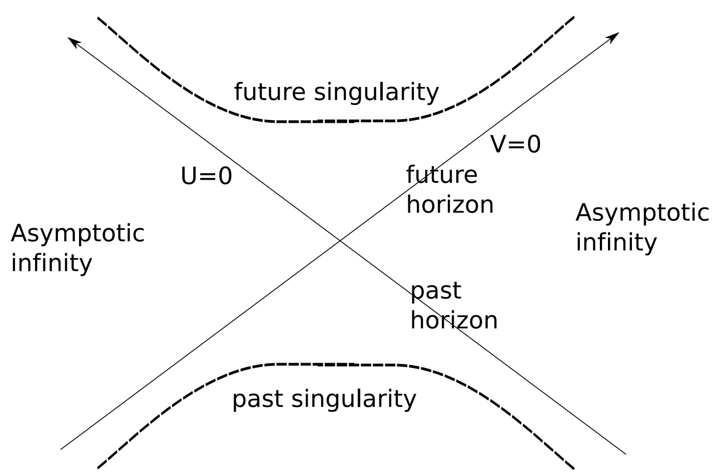

Figure B.1: The fully extended Schwarzschild geometry

The region outside the horizon was given by the coordinate range (B.17). Let us now see how we would extend the spacetime past the horizon, to reach the interior of the black hole. We let the metric continue to have the form (B.23), where $r(U, V)$ will continue to be given through (B.25). There is no problem with either equation at $r = 2M$. There will be a singularity at $r = 0$, which is a real singularity: the curvature diverges there, and we cannot remove this singularity with a coordinate transformation. From (B.25) we see that

$$r = 0 \;\leftrightarrow\; UV = 1 \tag{B.26}$$

We see that we can extend the coordinate range from the initial range (B.17) to all values of U, V satisfying $UV < 1$. This spacetime is called the 'extended black hole spacetime', and we depict it in fig.B.42. There is a 'future singularity' at $U > 0, V > 0, UV = 1$; if an observer decides to fall into the black hole then he will hit this singularity sometime in his future. But there is another singularity – the 'past singularity' at $U < 0, V < 0, UV = 1$. We will discuss the structure of this spacetime in more detail after drawing the Penrose diagram.

B.4 The Penrose diagram

The U, V coordinates cover all of our spacetime, but these coordinates do not have a bounded range. Thus if we try to draw the U, V space on a sheet of paper, we have to stop at a finite value of U, V, and we do not explicitly see the picture of how the 'points at infinity' border our spacetime. To bring these 'points at infinity' to a finite coordinate distance from the points in the interior of our spacetime, we make a conformal rescaling of the metric. Here the word 'conformal' means that at each point the metric is scaled by a number $g_{ab}(x) \to \Omega^2(x) g_{ab}(x)$, so that the angles between different directions at the point x do not change, and in particular null directions remain null directions. Such a rescaling will help us to understand the causal structure of the spacetime, including the behavior of 'infinity'.

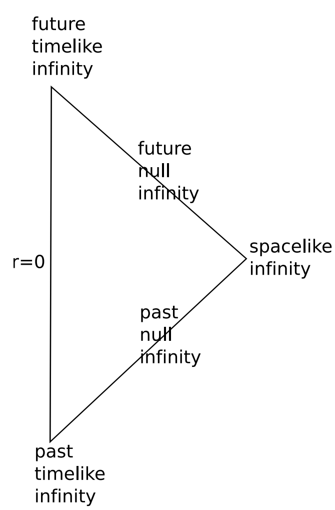

Figure B.2: Penrose diagram of Minkowski space

Let us first carry out this process for Minkowski spacetime; we will need this result anyway to describe part of the black hole spacetime when the black hole is made by 'collapse' of a shell. Minkowski spacetime is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{B.27}$$

Let us write

$$U = t + r, \quad V = t - r \tag{B.28}$$

getting

$$ds^2 = -dUdV + r^2 d\Omega^2 \tag{B.29}$$

where now the coordinates are U, V, θ, ϕ, and $r = \frac{1}{2}(U - V)$. Since

$$r = \frac{1}{2}(U - V) \geq 0 \tag{B.30}$$

we have the alllowed range

$$-\infty < U < \infty, \quad -\infty < V < \infty, \quad U \geq V \tag{B.31}$$

so we have an infinite coordinate range. Let us write

$$\tilde{U} = \tanh U, \quad \tilde{V} = \tanh V \tag{B.32}$$

so that

$$-1 < \tilde{U} < 1, \quad -1 < \tilde{V} < 1, \quad \tilde{U} \geq \tilde{V} \tag{B.33}$$

and the metric is

$$ds^2 = -[\frac{dU}{d\tilde{U}}\frac{dV}{d\tilde{V}}]d\tilde{U}\tilde{V} + r^2 d\Omega^2 \tag{B.34}$$

But

$$\frac{dU}{d\tilde{U}} = \operatorname{sech}^2 U = \frac{1}{1 - \tilde{U}^2}, \quad \frac{dV}{d\tilde{V}} = \operatorname{sech}^2 V = \frac{1}{1 - \tilde{V}^2} \tag{B.35}$$

so that

$$ds^2 = \frac{1}{(1 - \tilde{U}^2)(1 - \tilde{V}^2)}[-d\tilde{U}d\tilde{V} + r^2(1 - \tilde{U}^2)(1 - \tilde{V}^2)d\Omega^2] \tag{B.36}$$

So far we have just rewritten Minkowski spacetime in new coordinates, but now let us make a conformal transformation, defining a new metric

$$g'_{ab} = (1 - \tilde{U}^2)(1 - \tilde{V}^2)g_{ab} \tag{B.37}$$

Thus the new metric is

$$ds'^2 = -d\tilde{U}d\tilde{V} + r^2(1 - \tilde{U}^2)(1 - \tilde{V}^2)d\Omega^2 \tag{B.38}$$

Let us ignore the angular directions; since we have spherical symmetry there is no nontrivial structure in these directions, and the size if the angular sphere is not relevant for the main computations we are interested in. Thus we get

$$ds'^2 = -d\tilde{U}d\tilde{V} \tag{B.39}$$

with the coordinate range (B.33). The null directions are $\tilde{U} = U_0$ and $\tilde{V} = V_0$. This gives the Penrose diagram in fig.B.2.

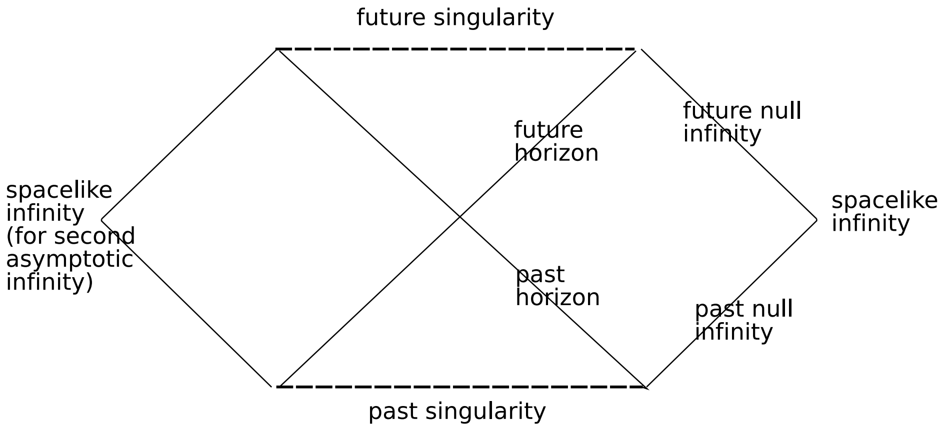

Figure B.3: Penrose diagram for the 'eternal Schwarzschild hole'

We now do a similar transformation (B.32),(B.37) for the black hole metric (B.23), getting

$$ds'^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} d\tilde{U} d\tilde{V} \qquad \text{(B.40)}$$

We have to be careful about the coordinate ranges though. The spacetime again ends at $r = 0$; this time there is a singularity there instead of a 'simple origin of coordinates'. But $r = 0$ is now given by solving $UV = 1$ which is

$$\tanh^{-1}\tilde{U}\ \tanh^{-1}\tilde{V} = 1 \qquad \text{(B.41)}$$

This is a curve in $\tilde{U}, \tilde{V}$ space, and points beyond this curve are not in the spacetime represented by the Penrose diagram, since they lie past the singularity. We draw the Penrose diagram in fig.B.3. The singularity runs along a curve from $\tilde{U} = 0, \tilde{V} = 1$ to $\tilde{U} = 1, \tilde{V} = 0$. Note that the causal structure of infinity is not changed by any other conformal rescaling of the metric at interior points of spacetime. Thus we can imagine a further rescaling which makes the singularity a straight line $\tilde{U} = 0, \tilde{V} = 1$ to $\tilde{U} = 1, \tilde{V} = 0$; this is easier to draw, and is typically what is done in figures. The essential property of the singularity we cannot change in the picture is that the singularity is *spacelike*; The constant r surface $r = 0$ is inside the horizon and so is spacelike instead of timelike.

B.5 The black hole formed by collapse

The black hole spacetime made above does not describe a realistic black hole made by collapse of a star. The spacetime we have found has a 'past singularity', which a collapsing star would not have, and also a second asymptotically flat region, which we cannot hope to produce simply by letting a star collapse in our starting spacetime. To get the correct spacetime for the collapsing star, note

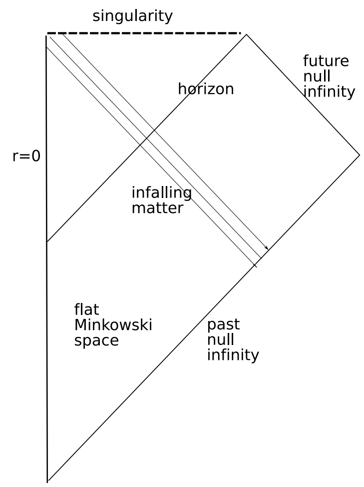

Figure B.4: Penrose diagram of the black hole made by collapse of a shell

that the metric inside a spherical shell is *flat Minkowski spacetime*. This follows by the Birkoff theorem, which says that a spherically symmetric vacuum solution to Einstein's equations must be a piece of the Schwarzscild geometry; since we have no source inside the shell, we must choose the geometry with $M = 0$, which is just Minkowski space. Thus inside the shell we take Minkowski spacetime, and outside the shell we must glue this to the black hole spacetime (using 'Israel matching conditions' across the shell). The resulting spacetime, shown in fig.B.4, does not have either the past singularity or the second asymptotically flat region.

To carry out this gluing, let us write the Kruskal and Minkowski coordinates in a more convenient way. We let all null coordinates have units of length. Note that the coordinates (B.16) were dimensionless. We now write

$$U_K = 4Me^{\frac{u}{4M}}, \quad V_K = -4Me^{-\frac{v}{4M}} \tag{B.42}$$

as our Kruskal coordinates. The black hole metric is then

$$ds^2_{BH} = -\frac{2M}{r}e^{-\frac{r}{2M}}dU_K dV_K + r^2 d\Omega^2 \tag{B.43}$$

The Minkowski coordiantes (B.28) are similarly given a label M

$$ds^2_M = -dU_M dV_M \tag{B.44}$$

We let the black hole be made by collapse of a null shell at

$$U_M = U_0 \tag{B.45}$$

We have to match metrics across this shell, since there is a coordinate system where the metric is continuous across the shell. The matching is at a constant value of U, and we set

$$U_K = U_0 \tag{B.46}$$

The metric functions can be made to agree by choosing an appropriate relation between the two V coordinates

$$V_K = V_K(V_M) \tag{B.47}$$

This function is to be found by matching

$$-\frac{2M}{r}e^{-\frac{r}{2M}}dU_K dV_K = -dU_M dV_M \tag{B.48}$$

Recall that

$$r = \frac{U_M - V_M}{2} \tag{B.49}$$

This gives

$$\frac{4M}{U_0 - V_M}e^{-\frac{U_0-V_M}{4M}}dV_K = dV_M \tag{B.50}$$

The solution is

$$V_K = \int dV_M \frac{U_0 - V_M}{4M}e^{\frac{U_0-V_M}{4M}} = -4M[\frac{U_0 - V_M}{4M}e^{\frac{U_0-V_M}{4M}} - e^{\frac{U_0-V_M}{4M}}] \tag{B.51}$$

where we have set an arbitrary additive constant to zero.

Recall that the natural coordinate at infinity is given by (B.42)

$$v = -4M\ln[-\frac{V_K}{4M}] = V_M - U_0 - 4M\ln(\frac{U_0 - V_M - 4M}{4M}) \tag{B.52}$$

Now recall that the horizon is $V_K = 0$, and the region outside the horizon is $V_K < 0$. We will be interested in particular in the null rays $V_K = -\epsilon$, where ϵ is small and positive; these are outgoing null rays just outside the horizon. From (B.42) we see that $V_K \to 0^-$ corresponds to $v \to \infty$. From (B.52) we see that we get $v \to -\infty$ if the argument of the log goes to 0^+, i.e.

$$V_M \to U_0 - 4M \equiv \bar{V}_M \tag{B.53}$$

Thus in our limit of interest we have

$$v \approx -4M\ln(\bar{V}_M - V_M) \tag{B.54}$$

So we see that the natural coordinate at infinity v is related nonlinearly to the natural coordinate at the horizon.

B.6 Field theory

We start with the lagrangian density

$$L = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{B.55}$$

We write

$$\phi(t,x) = \sum_{n=-\infty}^{\infty} \phi_n(t) e^{ik_n x} \tag{B.56}$$

with

$$k_n = \frac{2\pi n}{L} \tag{B.57}$$

Note that since ϕ is real,

$$\phi_n = \phi^*_{-n} \tag{B.58}$$

We write

$$\phi_n = \phi_n^R + i\phi_n^I \tag{B.59}$$

The Lagrangian becomes

$$L = \sum_{n=1}^{\infty}[L(\dot{\phi}_n^R)^2 - L\omega_n^2(\phi_n^R)^2] + \sum_{n=1}^{\infty}[L(\dot{\phi}_n^I)^2 - L\omega_n^2(\phi_n^I)^2] + \frac{L}{2}[(\dot{\phi}_0)^2 - \omega_0^2(\phi_0)^2] \tag{B.60}$$

where

$$\omega_n^2 = k_n^2 + m^2 \tag{B.61}$$

Thus we see that the quantum field is just a collection of harmonic oscillators.

B.7 Particle creation in Heisenberg representation

$$\hat{x} = \frac{1}{\sqrt{2\omega}} e^{-i\omega t}\hat{a} + \frac{1}{\sqrt{2\omega}} e^{i\omega t}\hat{a}^\dagger \tag{B.62}$$

$$(f,g) = -i[f\partial_t g^* - g^*\partial_t f] \tag{B.63}$$

Thus with

$$f = \frac{1}{\sqrt{2\omega}} e^{-i\omega t} \tag{B.64}$$

$$(f,f) = -i\frac{1}{2\omega}2i\omega = 1, \quad (f,f^*) = 0 \tag{B.65}$$

Now we write

$$\hat{x} = \frac{1}{\sqrt{2\omega}} e^{-i\omega t}\hat{a} + \frac{1}{\sqrt{2\omega}} e^{i\omega t}\hat{a}^\dagger = \frac{1}{\sqrt{2\omega'}} e^{-i\omega' t}\hat{b} + \frac{1}{\sqrt{2\omega'}} e^{i\omega' t}\hat{b}^\dagger \tag{B.66}$$

Doing $(., f)$ on both sides, we get $\hat{a}$ on the LHS, and on the RHS we get at $t = 0$.

$$(-i)\frac{1}{2\sqrt{\omega\omega'}}(i(\omega' + \omega)) = \frac{\omega + \omega'}{2\sqrt{\omega\omega'}} \equiv \alpha \tag{B.67}$$

for $\hat{b}$ and

$$(-i)\frac{1}{2\sqrt{\omega\omega'}}(i(\omega' - \omega)) = \frac{\omega' - \omega}{2\sqrt{\omega\omega'}} \equiv \beta \tag{B.68}$$

for $\hat{b}^\dagger$. Thus

So we get

$$\hat{a} = \alpha\hat{b} + \beta\hat{b}^\dagger \tag{B.69}$$

Then we get

$$|\psi>=|0>_a = Ce^{\frac{1}{2}\gamma b^\dagger b^\dagger}|0>_b \tag{B.70}$$

We find

$$\gamma = -\frac{\beta}{\alpha} \tag{B.71}$$

$$\gamma = -\frac{\omega' - \omega}{\omega' + \omega} \tag{B.72}$$

B.8 Particle creation in curved space

The story of Hawking radiation really begins with the understanding of particle creation in curved spacetime. (For reviews see [?].) Particles are described in terms of an underlying quantum field, say a scalar field ϕ. We can write a covariant action for this field, and do a path integral. But how do we define particles? In flat space we expand the field operator as

$$\hat{\phi} = \sum_{\vec{k}} \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega}}\left(\hat{a}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}-i\omega t} + \hat{a}^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}+i\omega t}\right) \tag{B.73}$$

where V is the volume of the spatial box where we have taken the field to live, and $\omega = \sqrt{|\vec{k}|^2 + m^2}$ for a field with mass m. The vacuum is the state annihilated by all the $\hat{a}$

$$\hat{a}_{\vec{k}}|0\rangle = 0 \tag{B.74}$$

and the $\hat{a}^\dagger_{\vec{k}}$ create particles.

In *curved* spacetime, on the other hand, there is no canonical definition of particles. We can choose any coordinate t for time, and decompose the field into positive and negative frequency modes with respect to this time t. Let the positive frequency modes be called $f(x)$; then their complex conjugates give negative frequency modes $f^*(x)$. The field operator can be expanded as

$$\hat{\phi}(x) = \sum_n \left(\hat{a}_n f_n(x) + \hat{a}^\dagger_n f^*_n(x)\right) \tag{B.75}$$

Then we can define a vacuum state as one that is annihilated by all the annihilation operators

$$\hat{a}_n|0\rangle_a = 0 \tag{B.76}$$

The creation operators generate particles; for example a 1-particle state would be

$$|\psi\rangle = \hat{a}^\dagger_n|0\rangle_a \tag{B.77}$$

We have added the subscript a to the vacuum state to indicate that the vacuum is defined with respect to the operators $\hat{a}_n$. But since there is no unique choice of the time coordinate t, we can choose a different one $\tilde{t}$. We will then have a different set of positive and negative frequency modes, and an expansion

$$\hat{\phi}(x) = \sum_n \left(\hat{b}_n h_n(x) + \hat{b}_n^\dagger h_n^*(x)\right) \tag{B.78}$$

Now the vacuum would be defined as

$$\hat{b}_n|0\rangle_b = 0 \tag{B.79}$$

and the $\hat{b}_n^\dagger$ would create particles.

The main point now is that a person using the operators $\hat{a}, \hat{a}^\dagger$ would think that $|0\rangle_a$ was a vacuum, but he would not think that the state $|0\rangle_b$ was a vacuum – he would find it to contain particles of the type created by the $\hat{a}_n^\dagger$. Let us see how one finds exactly how many $\hat{a}^\dagger$ particles there are in the state $|0\rangle_b$. The mode functions f_n are normalized using an inner product defined as follows. Take any spacelike hypersurface, with volume element $d\Sigma^\mu$ (thus the vector $d\Sigma^\mu$ points normal to the hypersurface and has a value equal to the volume of the surface element). Then

$$(f, g) \equiv -i \int d\Sigma^\mu \left(f\partial_\mu g^* - g^*\partial_\mu f\right) \tag{B.80}$$

Under this inner product we will have

$$(f_m, f_n) = \delta_{mn}, \quad (f_m, f_n^*) = 0, \quad (f_m^*, f_n^*) = -\delta_{mn} \tag{B.81}$$

Now from the two different expansions of $\hat{\phi}$ we have

$$\sum_n \left(\hat{a}_n f_n(x) + \hat{a}_n^\dagger f_n^*(x)\right) = \sum_n \left(\hat{b}_n h_n(x) + \hat{b}_n^\dagger h_n^*(x)\right) \tag{B.82}$$

Taking the inner product with f_m on each side, we get

$$\hat{a}_m = \sum_n (h_n, f_m)\hat{b}_n + \sum_n (h_n^*, f_m)\hat{b}_n^\dagger \equiv \sum_n \alpha_{mn}\hat{b}_n + \sum_n \beta_{mn}\hat{b}_n^\dagger \tag{B.83}$$

Thus the vacuum $|0\rangle_a$ satisfies

$$0 = \hat{a}_m|0\rangle_a = \left(\sum_n \alpha_{mn}\hat{b}_n + \sum_n \beta_{mn}\hat{b}_n^\dagger\right)|0\rangle_a \tag{B.84}$$

Let us see how to solve this equation. Suppose we had just one mode, with a relation

$$(b + \gamma b^\dagger)|0\rangle_a = 0 \tag{B.85}$$

The solution to this equation is of the form

$$|0\rangle_a = Ce^{\mu \hat{b}^\dagger \hat{b}^\dagger}|0\rangle_b \tag{B.86}$$

where C is a normalization constant and μ is a number that we have to determine. Expand the exponential in a power series

$$e^{\mu \hat{b}^\dagger \hat{b}^\dagger} = \sum_n \frac{\mu^n}{n!} (\hat{b}^\dagger \hat{b}^\dagger)^n \tag{B.87}$$

With a little effort using the commutator $[\hat{b}, \hat{b}^\dagger] = 1$ we find that

$$\hat{b}(\hat{b}^\dagger \hat{b}^\dagger)^n = (\hat{b}^\dagger \hat{b}^\dagger)^n \hat{b} + 2n \hat{b}^\dagger (\hat{b}^\dagger \hat{b}^\dagger)^{n-1} \tag{B.88}$$

Putting this in the series for the exponential, we find that

$$\hat{b} e^{\mu \hat{b}^\dagger \hat{b}^\dagger} |0\rangle_b = 2\mu \hat{b}^\dagger e^{\mu \hat{b}^\dagger \hat{b}^\dagger} |0\rangle_b \tag{B.89}$$

Looking at (B.85) we see that we should choose $\mu = -\frac{\gamma}{2}$, and we get

$$|0\rangle_a = C e^{-\frac{\gamma}{2} \hat{b}^\dagger \hat{b}^\dagger} |0\rangle_b \tag{B.90}$$

This state has the form

$$|0\rangle_a = C|0\rangle_b + C_2 \hat{b}^\dagger \hat{b}^\dagger |0\rangle_b + C_4 \hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger |0\rangle_b + \ldots \tag{B.91}$$

so it looks like a part that is the b vacuum, a part that has two particles of type b, a part with four such particles, and so on.

Returning to our full equation (B.84) we have the solution

$$|0\rangle_a = C e^{-\frac{1}{2} \sum_{m,n} \hat{b}^\dagger_m \gamma_{mn} \hat{b}^\dagger_n} |0\rangle_b \tag{B.92}$$

where the matrix γ is symmetric and is given by

$$\gamma = \frac{1}{2} \left(\alpha^{-1} \beta + (\alpha^{-1} \beta)^T \right) \tag{B.93}$$

Thus we see that

$$\hat{b}_k + e^{-4\pi M k} \hat{c}^\dagger_k \tag{B.94}$$

is made of annihilation operators $\hat{a}_K$, and so will kill the vacuum. The state is therefore

$$\prod_{k>0} e^{-e^{-4\pi M k} b^\dagger_k c^\dagger_k} |0\rangle_b \otimes |0\rangle_c \tag{B.95}$$

For each fourier mode k we get a state that is entangled between the b, c spaces

$$e^{-e^{-4\pi M k} b^\dagger_k c^\dagger_k} |0\rangle_b \otimes |0\rangle_c = |0\rangle_b \otimes |0\rangle_c - e^{-4\pi M k} b^\dagger_k c^\dagger_k |0\rangle_b \otimes |0\rangle_c + \frac{1}{2!} e^{-8\pi M k} b^\dagger_k b^\dagger_k c^\dagger_k c^\dagger_k |0\rangle_b \otimes |0\rangle_c \ldots \tag{B.96}$$

This is

$$|0\rangle_b \otimes |0\rangle_c - e^{-4\pi M k} |1\rangle_b \otimes |1\rangle_c + e^{-8\pi M k} |2\rangle_b \otimes |2\rangle_c \ldots \tag{B.97}$$

B.9 Density matrices

Suppose we have a state

$$\sum_n C_n |n\rangle_b \otimes |n\rangle_c \tag{B.98}$$

If we trace over c, then we get the density matrix

$$\rho = \sum_n |C_n|^2 |n\rangle_{bb}\langle n| \tag{B.99}$$

Suppose

$$C_n = (-1)^n e^{-4\pi Mkn} \tag{B.100}$$

Then we have the probability of getting n pairs down by a factor

$$e^{-4\pi Mkn} \tag{B.101}$$

This should equal

$$e^{-\frac{kn}{T}} \tag{B.102}$$

Thus

$$T = \frac{1}{8\pi M} \tag{B.103}$$

Restoring Newton's constant, we get $M \to GM$, getting

$$T = \frac{1}{8\pi GM} \tag{B.104}$$

B.10 Entangled states

Suppose we have a 2 state system, with state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2] \tag{B.105}$$

Suppose we want to trace over system 2, getting a density matrix for system 1. First we write

$$|\psi\rangle\langle\psi| = \frac{1}{2}[|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2][{}_1\langle\uparrow| \otimes {}_2\langle\downarrow| + {}_1\langle\downarrow| \otimes {}_2\langle\uparrow|] \tag{B.106}$$

This density matrix is made of 4 terms, which we write explicitly

$$|\psi\rangle\langle\psi| = \frac{1}{2}[|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \; {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| \tag{B.107}$$

$$+|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \; {}_1\langle\downarrow| \otimes {}_2\langle\uparrow| \tag{B.108}$$

$$+|\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \; {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| \tag{B.109}$$

$$+|\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \; {}_1\langle\downarrow| \otimes {}_2\langle\uparrow|] \tag{B.110}$$

In taking the trace over system 2, we keep only the terms where the state of system 2 is the same in the bra and in the ket, and write down only the system 1 states for these terms

$$\rho = \frac{1}{2}[|\uparrow\rangle_1\ {}_1\langle\uparrow| \ + \ |\downarrow\rangle_1\ {}_1\langle\downarrow|] \tag{B.111}$$

Let the basis of ket states be $\{|\uparrow\rangle_1, |\downarrow\rangle_1\}$. Then we can write

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{B.112}$$

Now we compute the entropy

$$S = -tr[\rho \ln \rho] \tag{B.113}$$

This works out to

$$-2(\frac{1}{2}\ln(\frac{1}{2})) = \ln 2 \tag{B.114}$$

Thus the entanglement entropy of the first system with the second is $\ln 2$.

B.11 The problem with black hole disappearance

Suppose we start with an entangled system, and one part of it disappears. Then we cannot write down any sensible wavefunction for the part that remains. To see this, start with the following entangled system, and try to write down a state for the first part alone

$$\frac{1}{\sqrt{2}}[(+)(-) + (-)(+)] \to \frac{1}{\sqrt{2}}[(-) + (+)] \tag{B.115}$$

But now we can change the basis of the part that has disappeared

$$(+') = e^{i\alpha}(+), (-') = e^{-i\alpha}(-) \tag{B.116}$$

Now we would deduce the state

$$\frac{1}{\sqrt{2}}[e^{-i\alpha}(+')(-) + e^{i\alpha}(-')(+)] \to \frac{1}{\sqrt{2}}[e^{-i\alpha}(-) + e^{i\alpha}(+)] \tag{B.117}$$

which is a different state. For example if we let the two states be the up and down z spins of a spin 1/2 particle, then first attempt would give a spin polarized in direction x, while in the second attempt, with $\alpha = \frac{\pi}{2}$, we would get spin in the direction y.

Bibliography