1 The Classical String

A point particle sweeps out a 1-dimensional worldline in spacetime. A string sweeps out a 2-dimensional *worldsheet*. The action of the point particle is given by the length of the worldline

$$S = -m \int ds \tag{1}$$

where m is the mass of the particle, with units $\frac{1}{L}$. The action of the fundamental string is given by the area of the worldsheet

$$S = -T \int dA \tag{2}$$

where T is the 'tension' of the string, with units $\frac{1}{L^2}$.

Note that the dynamics of such a 'fundamental' string is different from that of a string that we encounter in the everyday world. The latter string is made of atoms, joined together by chemical bonds, and it can have both transverse and longitudinal vibrations. In a transverse deformation the string occupies different points of spacetime, while in a longitudinal deformation the string profile does not change – the atoms change their separations to move to different positions along the string. The fundamental string is not 'made up' of atoms or anything else, since it is 'fundamental'. Thus it does not have longitudinal vibrations, only transverse ones.

There is a natural way to encounter the dynamics of such a fundamental string in the classical world. Consider a solitonic string: this could be a cosmic string, or a vortex in a superfluid, or more generally any 1-dimensional defect in a fluid. When the defect moves to a new place we get a transverse deformation of the string, but there are no *longitudinal* deformations, since the string is not made of 'atoms'; rather it is just the set of points where the surrounding medium had a topological defect.

The absence of longitudinal modes for the fundamental string will be very important for the way that we will describe its dynamics, and it will have very important consequences when we study black holes.

1.1 Equations of motion

To obtain dynamical equations from the actions (1),(2) we need to write these actions in terms of fields over the worldline or worldsheet. To do this we will need to introduce coordinates on the worldline/worldsheet. There are no natural coordinates here, so we will have to make an arbitrary choice. Let us start with the point particle.

1.1.1 The point particle in flat space

Let the points along the wordline be labeled by a coordinate τ. The configuration of the string is then given by specifying the functions $X^\mu(\tau),\ \mu = 0, 1, \ldots D-1$. Then the action (1) is

$$S = -m \int ds = -m \int \sqrt{-dX^\mu dX_\mu} = -m \int \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}\, d\tau \tag{3}$$

We have assumed that the trajectory is timelike. (Our signature is $- + + \ldots +$)

The equation of motion for X^μ is

$$\frac{d}{d\tau}\left[\frac{\frac{dX^\mu}{d\tau}}{\sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}}\right] \ = \ 0 \tag{4}$$

Expanded out, this is

$$\frac{d^2X^\mu}{d\tau^2} - \frac{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}{(\frac{dX^\lambda}{d\tau}\frac{dX_\lambda}{d\tau})}\frac{d^2X_\nu}{d\tau^2} \ = \ [\eta^{\mu\nu} - \frac{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}{(\frac{dX^\lambda}{d\tau}\frac{dX_\lambda}{d\tau})}]\frac{d^2X_\nu}{d\tau^2} \ = \ 0 \tag{5}$$

The term in box brackets is a projector with rank $D-1$ since it has a null eigenvector

$$[\eta^{\mu\nu} - \frac{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}{(\frac{dX^\lambda}{d\tau}\frac{dX_\lambda}{d\tau})}]\ \frac{dX_\nu}{d\tau} \ = \ 0 \tag{6}$$

Thus we cannot use (5) to find all the D expressions $\frac{d^2X^\mu}{d\tau^2}$; one will be left undetermined. This is as it should be, since τ was an arbitrary parameter, and so we could not have found all the functions $X^\mu(\tau)$ uniquely. We can choose τ in many ways. In the 'static' gauge, we write

$$\tau = X^0 \tag{7}$$

so that the value of τ at any point along the worldline is given by the value of the time coordinate X^0 at that point of the worldline. Another gauge choice is

$$-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} \ = \ 1 \tag{8}$$

With this choice, the equations (4) simplify

$$\frac{d^2X^\mu}{d\tau^2} \ = \ 1 \tag{9}$$

so that the X^μ are linear functions of τ in this gauge

$$X^\mu = a^\mu + b^\mu\tau \tag{10}$$

But these linear functions must be chosen to satisfy the constraint (8), so we must have

$$b^\mu b_\mu = -1 \tag{11}$$

Note that (8) can be written as

$$g^{ind}_{\tau\tau} = -1 \tag{12}$$

where $g^{ind}_{\tau\tau}$ is the 'induced metric' on the particle worldline. (The induced metric gives the length of an infinitesimal segment of the worldline by measuring the separation of its endpoints in the ambient D dimensional spacetime.)

1.1.2 The string in flat space

We need the analog of (3). Note that for the point particle case we have

$$\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} \;=\; g^{ind}_{\tau\tau} \tag{13}$$

where $g^{ind}_{\tau\tau}$ is the 'induced metric' on the worldline. Thus S is $(-m)$ times the length of the worldline measured in this induced metric

$$S = -m\int\sqrt{-g^{ind}}\;d\tau \tag{14}$$

For the string, we will need two coordinates to describe the worldsheet. We assume that one direction along the string worldsheet is timelike, just like for the worldline; we will call this coordinate τ. The other will then have to be spacelike, and we will call it σ. We will write $\xi^a = (\xi^0, \xi^1) = (\tau, \sigma)$. The configuration of the string worldsheet is parametrized by giving the functions

$$X^\mu(\tau,\sigma), \quad \mu = 0, 1, \dots (D-1) \tag{15}$$

The metric induced on the worldsheet is

$$g^{ind}_{ab} = \begin{pmatrix} g^{ind}_{\tau\tau} & g^{ind}_{\tau\sigma} \\ g^{ind}_{\sigma\tau} & g^{ind}_{\sigma\sigma}\end{pmatrix} = \begin{pmatrix} \frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\tau} & \frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\sigma} \\ \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau} & \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\sigma}\end{pmatrix} \tag{16}$$

Thus

$$\begin{aligned} S \;&=\; -T\int\sqrt{-g^{ind}}\;d^2\xi \;=\; -T\int\sqrt{-g^{ind}_{\tau\tau}g^{ind}_{\sigma\sigma} + (g^{ind}_{\tau\sigma})^2}\;d\tau d\sigma \\ &=\; -T\int d\tau d\sigma\sqrt{-\left(\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\tau}\right)\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\sigma}\right) + \left(\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\sigma}\right)^2} \end{aligned} \tag{17}$$

Let us write

$$(X^\mu)^{\dot{}} = \frac{\partial X^\mu}{\partial\tau}, \qquad (X^\mu)' = \frac{\partial X^\mu}{\partial\sigma} \tag{18}$$

Then the Euler-Lagrange equations are

$$\frac{\partial}{\partial\tau}\left[\frac{-(X'\cdot X')(X^\mu)^{\dot{}} + (\dot X\cdot X')(X^\mu)'}{\sqrt{-(\dot X\cdot\dot X)(X'\cdot X') + (\dot X\cdot X')^2}}\right] + \frac{\partial}{\partial\sigma}\left[\frac{-(\dot X\cdot\dot X)(X^\mu)' + (\dot X\cdot X')(X^\mu)^{\dot{}}}{\sqrt{-(\dot X\cdot\dot X)(X'\cdot X') + (\dot X\cdot X')^2}}\right] \;=\; 0 \tag{19}$$

These equations look complicated, but just as for the point particle case, we can choose a gauge to simplify the equations. We can choose any two coordinate functions to parametrize the string world sheet, so we have the freedom of choosing two functions of two variables each:

$$\tilde\tau = \tilde\tau(\tau,\sigma), \qquad\qquad \tilde\sigma = \tilde\sigma(\tau,\sigma) \tag{20}$$

Thus the field equations (19) could not have determined all D functions $X^\mu(\tau,\sigma)$, but only $D-2$ of them. Thus we can 'gauge fix' two combinations of the variables $X^\mu(\tau,\sigma)$. The 'static gauge' is suited to the case where the string is stretched along one direction, say X^1, and then undergoes small transverse vibrations. In this gauge we set

$$X^0 = \tau, \qquad X^1 = \sigma \tag{21}$$

and then the field equations determine the remaining $(D-2)$ variables $X^2 \ldots X^{D-1}$. We will return to the study of this gauge later. Another gauge choice is the analog of (8), which we rewrote as (12). This time there are three independent components of the induced metric on the worldsheet $g^{ind}_{\tau\tau}, g^{ind}_{\sigma\sigma}, g^{ind}_{\tau\sigma} = g^{ind}_{\sigma\tau}$. We can set two combinations to chosen values, by using the two freedoms of coordinates (20). One condition will be $g^{ind}_{\tau\sigma} = 0$

$$\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma} = 0 \tag{22}$$

The other will set $g^{ind}_{\tau\tau} + g^{ind}_{\sigma\sigma} = 0$

$$\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau} + \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \sigma} = 0 \tag{23}$$

The induced metric then has the form

$$\begin{pmatrix} g^{ind}_{\tau\tau} & g^{ind}_{\tau\sigma} \\ g^{ind}_{\sigma\tau} & g^{ind}_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} -e^{2\rho} & 0 \\ 0 & e^{2\rho} \end{pmatrix} = e^{2\rho}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

where $\rho = \rho(\tau, \sigma)$. Thus the induced metric on the world sheet has been brought to a 'conformal factor' $e^{2\rho}$ times the standard 2-dimensional Minkowski metric.

In the abbreviated notation (18) we have

$$\dot{X} \cdot X' = 0, \qquad \dot{X} \cdot \dot{X} = -e^{2\rho}, \qquad X' \cdot X' = e^{2\rho} \tag{25}$$

We then see that the quantity in the square roots in (19) simplifies

$$-(\dot{X} \cdot \dot{X})(X' \cdot X') + (\dot{X} \cdot X')^2 = e^{4\rho} \tag{26}$$

and (19) becomes

$$[\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2}]X^\mu = 0 \tag{27}$$

These equations are the analog of (9), and can be solved almost as easily. But an arbitrary solution of these equations does not give an allowed motion of the string. We must ensure that the chosen solution satisfies (22),(23), which are the analogs of (8). It is quite a bit harder to impose the constraints in the string case, and so the allowed motions are not as obvious as in the particle case.

1.1.3 Simple examples

Let us make some allowed motions for the string. First we try

$$X^0 = \tau, \quad X^1 = \tau \tag{28}$$

with all other X^μ vanishing. Note that this solution satisfies the harmonic equations (27) and the constraints (22),(23). Since no X^μ depends on σ, The string has no oscillations, and behaves like a point particle. This particle is moving in the direction X^1 with the speed of light. Thus we have recovered the fact that the classical string, in its ground state, describes massless point particles.

Now let us try to make a massive string state; i.e., a state where the string is oscillating, and thus represents a massive particle. Since it is a massive particle, we can take it to be at rest. We expect that τ will be along the timelike direction of the string, so we set

$$X^0 = \alpha\tau \tag{29}$$

where α is a constant. This satisfies (27). For the vibrations, we set

$$X^1 = a\cos(\tau-\sigma), \quad X^2 = a\sin(\tau-\sigma), \quad X^3 = a\cos(\tau+\sigma), \quad X^4 = a\sin(\tau+\sigma) \tag{30}$$

with all other X^μ vanishing. All the X^μ satisfy the harmonic condition (27), and one finds that with

$$\alpha = 2a \tag{31}$$

the constraints (22),(23) are satisfied. Note that we needed both right and left moving waves on the string (these are functions of $\tau-\sigma$ and $\tau+\sigma$ respectively); otherwise we cannot satisfy (22).

1.2 Using an auxiliary metric

While we have solved the classical motion of the string, our final goal is to quantize the string. The 'area action' (17) is very nonlinear in the dynamical variables X^μ, so if we do a path integral over the X^μ then the measure will be a complicated function of X^μ. Thus we seek another action which will give the same classical motions, but will have a simpler measure. Such an action can be obtained by using an auxiliary variable – the metric on the worldsheet.

Thus we will take our string world sheet and imagine that it is described not only by its embedding in spacetime $X^\mu(\tau,\sigma)$ but also by a metric on the worldsheet

$$g_{ab} = \begin{pmatrix} g_{\tau\tau} & g_{\tau\sigma} \\ g_{\sigma\tau} & g_{\sigma\sigma} \end{pmatrix} \tag{32}$$

Note that this metric g_{ab} is an independent variable, and is thus fundamentally different from the induced metric g^{ind}_{ab} given in eq. (16); the induced metric was just a functional of the X^μ. But we will find that the classical equations of motion set g_{ab} equal to the g^{ind}_{ab} determined by the X^μ.

For the most part the computations using an auxiliary metric will work the same way for any brane, so we start with a general p-brane. Thus the point particle is a 0-brane with a 1-dimensional worldline, the string is a 1-brane with a 2-dimensional worldsheet, and the general p-brane is a p-dimensional spatial object with a $(p+1)$-dimensional worldvolume. The dynamical variables are now $g_{ab}(\xi), X^\mu(\xi)$, where $\xi^a, a = 0, 1, \ldots p$ are the coordinates on the worldvolume.

With the metric g_{ab} we can write a simple quadratic action for the X^μ

$$S_X = -T\int d^{p+1}\xi\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{33}$$

where we again set the constant in front to make S_X have no units. We can also write a 'cosmological constant' term using the metric

$$S_g = \alpha\int d^{p+1}\xi\sqrt{-g} \tag{34}$$

so the total action is

$$S = S_X + S_g \tag{35}$$

First consider the variations with respect to the g_{ab}. A basic identity tells us how to vary a determinant

$$\delta \det M = (\det M) M^{-1}_{ab} (\delta M)_{ba} = -(\det M) M_{ab} (\delta M^{-1}_{ba}) \tag{36}$$

This gives

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab} \tag{37}$$

and we find the equation

$$\frac{\delta S}{\delta g^{ab}} = -T[-\frac{1}{2} g_{ab} (\partial_c X^\mu \partial^c X_\mu) + \partial_a X^\mu \partial_b X_\mu] - \frac{\alpha}{2} g_{ab} = 0 \tag{38}$$

We find that the induced metric is proportional to the metric g_{ab}

$$g^{ind}_{ab} \equiv \partial_a X^\mu \partial_b X_\mu = [\frac{1}{2}(\partial_c X^\mu \partial^c X_\mu) - \frac{\alpha}{2T}]\ g_{ab} \tag{39}$$

To find α let us trace over a, b in (39)

$$(\partial_c X^\mu \partial^c X_\mu) = \frac{(p+1)}{2}(\partial_c X^\mu \partial^c X_\mu) - \frac{\alpha(p+1)}{2T} \tag{40}$$

For the string $p = 1$ we find that we must have

$$\alpha = 0 \tag{41}$$

so there is no cosmological constant term in the action. For $p \neq 1$ we find that we will get

$$(\partial_c X^\mu \partial^c X_\mu) = \frac{\alpha}{T}\frac{(p+1)}{(p-1)} \tag{42}$$

so the trace of the induced metric will be a constant when the equations of motion are satisfied.

We still have to look at the X^μ equations of motion. These come only from S_X, and we have

$$\frac{1}{\sqrt{-g}}\partial_a[\sqrt{-g} g^{ab} \partial_b X^\mu] = \Box_g X^\mu = 0 \tag{43}$$

so that each X^μ satisfies the free wave equation on the worldvolume equipped with the metric g_{ab}. But an arbitrary solution of these free wave equations will not be a solution; we must satisfy the constraint that the induced metric be proportional to g_{ab}

$$\partial_a X^\mu \partial_b X_\mu = w(x) g_{ab} \tag{44}$$

Tracing over a, b, we immediately see that this is equivalent to $\partial_a X^\mu \partial_b X_\mu = \frac{1}{2} g_{ab}(\partial_c X^\mu \partial^c X_\mu)$.

Thus a solution is given by a metric g_{ab}, and harmonic functions X^μ, satisfying constraints (44). Do all such sets of g_{ab}, X^μ represent different solutions? No, since we can change coordinates on the worldsheet according to (20), and this will change both the functions g_{ab} and X^μ. We can use such diffeomorphims to bring g_{ab} to a simple form

$$g_{ab} = e^{2\rho}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{45}$$

Then the wave equation (43) becomes

$$[\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}]X^\mu = 0 \tag{46}$$

and the constraints (44) can be written as

$$\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\sigma} = 0 \tag{47}$$

$$\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\tau} + \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\sigma} = 0 \tag{48}$$

Thus we seem to have arrived at the same solution as the one we obtained without introducing the auxiliary metric g_{ab}. But now we do have a good measure that we can use. For the initial variables g_{ab}, X^μ the measure was of the form

$$\int D[g]D_g[X] \tag{49}$$

where $D[g]$ is some complicated measure on the space of metrics, and $D_g[X]$ is the measure on the X^μ but this measure will depend in general on the metric g_{ab} on the surface. Now use the diffeomorphism freedom (20) to bring g_{ab} to the conformal form (45). The measure over metrics will get a Jacobian factor that will again depend on the metric. It turns out that the measure over X factorizes into a part that depends on ρ and a part that is independent of ρ. The action

$$S = -T\int d^2\xi\frac{1}{2}\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X_\mu = -T\int d\tau d\sigma\frac{1}{2}[\partial_\tau X^\mu\partial_\tau X_\mu - \partial_\sigma X^\mu\partial_\sigma X_\mu] \tag{50}$$

becomes independent of the metric. Thus the path integral over g_{ab} and X^μ factorizes, and each part can be studied explicitly. We will see all this in more detail later when computing loop amplitudes.