SOLUTIONS MANUAL

CHAPTER 1

1. The energy contained in a volume dV is

$$U(\nu,T)dV = U(\nu,T)r^2 dr \sin\theta d\theta d\varphi$$

when the geometry is that shown in the figure. The energy from this source that emerges through a hole of area dA is

$$dE(\nu,T) = U(\nu,T)dV \frac{dA\cos\theta}{4\pi r^2}$$

The total energy emitted is

$$\begin{aligned} dE(\nu,T) &= \int_0^{c\Delta t} dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi U(\nu,T)\sin\theta\cos\theta \frac{dA}{4\pi} \\ &= \frac{dA}{4\pi} 2\pi c\Delta t U(\nu,T) \int_0^{\pi/2} d\theta \sin\theta\cos\theta \\ &= \frac{1}{4} c\Delta t dA U(\nu,T) \end{aligned}$$

.

By definition of the emissivity, this is equal to $E\Delta t dA$. Hence

$$E(\nu,T) = \frac{c}{4} U(\nu,T)$$

2. We have

$$w(\lambda,T) = U(\nu,T)\,|d\nu/d\lambda| = U(\frac{c}{\lambda})\frac{c}{\lambda^2} = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

This density will be maximal when $dw(\lambda,T)/d\lambda = 0$. What we need is

$$\frac{d}{d\lambda}\left(\frac{1}{\lambda^5}\frac{1}{e^{A/\lambda}-1}\right) = (-5\frac{1}{\lambda^6} - \frac{1}{\lambda^5}\frac{e^{A/\lambda}}{e^{A/\lambda}-1}(-\frac{A}{\lambda^2}))\frac{1}{e^{A/\lambda}-1} = 0$$

Where $A = hc/kT$. The above implies that with $x = A/\lambda$, we must have

$$5 - x = 5e^{-x}$$

A solution of this is $x = 4.965$ so that

$$\lambda_{max}T = \frac{hc}{4.965k} = 2.898\times 10^{-3}\,m$$

In example 1.1 we were given an estimate of the sun's surface temperature as 6000 K. From this we get

$$\lambda_{max}^{sun} = \frac{28.98\times 10^{-4}\,mK}{6\times 10^{3}K} = 4.83\times 10^{-7}\,m = 483nm$$

3. The relationship is

$$h\nu = K + W$$

where K is the electron kinetic energy and W is the work function. Here

$$h\nu = \frac{hc}{\lambda} = \frac{(6.626\times 10^{-34}\,J.s)(3\times 10^{8}\,m/s)}{350\times 10^{-9}\,m} = 5.68\times 10^{-19}\,J = 3.55eV$$

With K = 1.60 eV, we get W = 1.95 eV

4. We use

$$\frac{hc}{\lambda_1} - \frac{hc}{\lambda_2} = K_1 - K_2$$

since W cancels. From ;this we get

$$h = \frac{1}{c}\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(K_1 - K_2) =$$
$$= \frac{(200\times 10^{-9}\,m)(258\times 10^{-9}\,m)}{(3\times 10^{8}\,m/s)(58\times 10^{-9}\,m)}\times(2.3-0.9)eV\times(1.60\times 10^{-19})J/eV$$
$$= 6.64\times 10^{-34}\,J.s$$

5. The maximum energy loss for the photon occurs in a head-on collision, with the photon scattered backwards. Let the incident photon energy be $h\nu$, and the backward-scattered photon energy be $h\nu'$. Let the energy of the recoiling proton be E. Then its recoil momentum is obtained from $E = \sqrt{p^2c^2 + m^2c^4}$. The energy conservation equation reads

$$h\nu + mc^2 = h\nu' + E$$

and the momentum conservation equation reads

$$\frac{h\nu}{c} = -\frac{h\nu'}{c} + p$$

that is

$$h\nu = -h\nu' + pc$$

We get $E + pc - mc^2 = 2h\nu$ from which it follows that

$$p^2c^2 + m^2c^4 = (2h\nu - pc + mc^2)^2$$

so that

$$pc = \frac{4h^2\nu^2 + 4h\nu mc^2}{4h\nu + 2mc^2}$$

The energy loss for the photon is the kinetic energy of the proton
$K = E - mc^2$. Now $h\nu$ = 100 MeV and mc^2= 938 MeV, so that

$$pc = 182 MeV$$

and

$$E - mc^2 = K = 17.6 MeV$$

6. Let $h\nu$ be the incident photon energy, $h\nu'$ the final photon energy and p the outgoing electron momentum. Energy conservation reads

$$h\nu + mc^2 = h\nu' + \sqrt{p^2c^2 + m^2c^4}$$

We write the equation for momentum conservation, assuming that the initial photon moves in the x –direction and the final photon in the y-direction. When multiplied by c it read

$$\mathbf{i}(h\nu) = \mathbf{j}(h\nu') + (\mathbf{i}p_x c + \mathbf{j}p_y c)$$

Hence $p_x c = h\nu; p_y c = -h\nu'$. We use this to rewrite the energy conservation equation as follows:

$$(h\nu + mc^2 - h\nu')^2 = m^2c^4 + c^2(p_x^2 + p_y^2) = m^2c^4 + (h\nu)^2 + (h\nu')^2$$

From this we get

$$h\nu' = h\nu\left(\frac{mc^2}{h\nu + mc^2}\right)$$

We may use this to calculate the kinetic energy of the electron

$$K = h\nu - h\nu' = h\nu\left(1 - \frac{mc^2}{h\nu + mc^2}\right) = h\nu\frac{h\nu}{h\nu + mc^2}$$

$$= \frac{(100keV)^2}{100keV + 510keV} = 16.4keV$$

Also

$$\mathbf{p}c = \mathbf{i}(100keV) + \mathbf{j}(-83.6keV)$$

which gives the direction of the recoiling electron.

7. The photon energy is

$$h\nu = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34}J.s)(3\times10^{8}m/s)}{3\times10^{6}\times10^{-9}m} = 6.63\times10^{-17}J$$

$$= \frac{6.63\times10^{-17}J}{1.60\times10^{-19}J/eV} = 4.14\times10^{-4}MeV$$

The momentum conservation for collinear motion (the collision is head on for maximum energy loss), when squared, reads

$$\left(\frac{h\nu}{c}\right)^2 + p^2 + 2\left(\frac{h\nu}{c}\right)p\eta_i = \left(\frac{h\nu'}{c}\right)^2 + p'^2 + 2\left(\frac{h\nu'}{c}\right)p'\eta_f$$

Here $\eta_i = \pm1$, with the upper sign corresponding to the photon and the electron moving in the same/opposite direction, and similarly for η_f. When this is multiplied by c^2 we get

$$(h\nu)^2 + (pc)^2 + 2(h\nu)pc\eta_i = (h\nu')^2 + (p'c)^2 + 2(h\nu')p'c\eta_f$$

The square of the energy conservation equation, with E expressed in terms of momentum and mass reads

$$(h\nu)^2 + (pc)^2 + m^2c^4 + 2Eh\nu = (h\nu')^2 + (p'c)^2 + m^2c^4 + 2E'h\nu'$$

After we cancel the mass terms and subtracting, we get

$$h\nu(E - \eta_i pc) = h\nu'(E' - \eta_f p'c)$$

From this can calculate $h\nu'$ and rewrite the energy conservation law in the form

$$E - E' = h\nu\left(\frac{E - \eta_i pc}{E' - p'c\eta_f} - 1\right)$$

The energy loss is largest if $\eta_i = -1; \eta_f = 1$. Assuming that the final electron momentum is not very close to zero, we can write $E + pc = 2E \quad and \quad E' - p'c = \frac{(mc^2)^2}{2E'}$ so that

$$E - E' = h\nu\left(\frac{2E \times 2E'}{(mc^2)^2}\right)$$

It follows that $\frac{1}{E'} = \frac{1}{E} + 16h\nu$ with everything expressed in MeV. This leads to E' =(100/1.64)=61 MeV and the energy loss is 39MeV.

8.We have λ' = 0.035 x 10^{-10} m, to be inserted into

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos 60^0) = \frac{h}{2m_e c} = \frac{6.63 \times 10^{-34} J.s}{2 \times (0.9 \times 10^{-30} kg)(3 \times 10^8 m/s)} = 1.23 \times 10^{-12} m$$

Therefore $\lambda = \lambda'$ = (3.50-1.23) x 10^{-12} m = 2.3 x 10^{-12} m.

The energy of the X-ray photon is therefore

$$h\nu = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34} J.s)(3 \times 10^8 m/s)}{(2.3 \times 10^{-12} m)(1.6 \times 10^{-19} J/eV)} = 5.4 \times 10^5 eV$$

9. With the nucleus initially at rest, the recoil momentum of the nucleus must be equal and opposite to that of the emitted photon. We therefore have its magnitude given by $p = h\nu/c$, where $h\nu = 6.2 \quad MeV$. The recoil energy is

$$E = \frac{p^2}{2M} = h\nu\frac{h\nu}{2Mc^2} = (6.2MeV)\frac{6.2MeV}{2 \times 14 \times (940MeV)} = 1.5 \times 10^{-3} MeV$$

10. The formula $\lambda = 2a\sin\theta/n$ implies that $\lambda/\sin\theta \le 2a/3$. Since $\lambda = h/p$ this leads to $p \ge 3h/2a\sin\theta$, which implies that the kinetic energy obeys

$$K = \frac{p^2}{2m} \ge \frac{9h^2}{8ma^2\sin^2\theta}$$

Thus the minimum energy for electrons is

$$K = \frac{9(6.63 \times 10^{-34} J.s)^2}{8(0.9 \times 10^{-30} kg)(0.32 \times 10^{-9} m)^2(1.6 \times 10^{-19} J/eV)} = 3.35eV$$

For Helium atoms the mass is $4(1.67\times10^{-27}kg)/(0.9\times10^{-30}kg) = 7.42\times10^{3}$ larger, so that

$$K = \frac{33.5eV}{7.42\times10^{3}} = 4.5\times10^{-3}eV$$

11. We use $K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$ with $\lambda = 15 \times 10^{-9}$ m to get

$$K = \frac{(6.63\times10^{-34}Js)^2}{2(0.9\times10^{-30}kg)(15\times10^{-9}m)^2(1.6\times10^{-19}J/eV)} = 6.78\times10^{-3}eV$$

For $\lambda = 0.5$ nm, the wavelength is 30 times smaller, so that the energy is 900 times larger. Thus K=6.10 eV.

12. For a circular orbit of radius r, the circumference is $2\pi r$. If n wavelengths λ are to fit into the orbit, we must have $2\pi r = n\lambda = nh/p$. We therefore get the condition

$$pr = nh/2\pi = n\hbar$$

which is just the condition that the angular momentum in a circular orbit is an integer in units of $\hbar$.

13. We have $a = n\lambda/2\sin\theta$. For $n = 1$, $\lambda = 0.5 \times 10^{-10}$ m and $\theta = 5^{o}$. we get $a = 2.87 \times 10^{-10}$ m. For $n = 2$, we require $\sin\theta_2 = 2\sin\theta_1$. Since the angles are very small, $\theta_2 = 2\theta_1$. So that the angle is 10^{o}.

14. The relation $F = ma$ leads to $mv^2/r = m\omega r$ that is, $v = \omega r$. The angular momentum quantization condition is $mvr = n\hbar$, which leads to $m\omega r^2 = n\hbar$. The total energy is therefore

$$E = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2r^2 = m\omega^2r^2 = n\hbar\omega$$

The analog of the Rydberg formula is

$$\nu(n \to n') = \frac{E_n - E_{n'}}{h} = \frac{\hbar\omega(n-n')}{h} = (n-n')\frac{\omega}{2\pi}$$

The frequency of radiation in the classical limit is just the frequency of rotation $\nu_{cl} = \omega/2\pi$ which agrees with the quantum frequency when $n - n' = 1$. When the selection rule $\Delta n = 1$ is satisfied, then the classical and quantum frequencies are the same *for all n*.

15. With $V(r) = V_0\,(r/a)^k$, the equation describing circular motion is

$$m\frac{v^2}{r} = \left|\frac{dV}{dr}\right| = \frac{1}{r}kV_0\left(\frac{r}{a}\right)^k$$

so that

$$v = \sqrt{\frac{kV_0}{m}}\left(\frac{r}{k}\right)^{k/2}$$

The angular momentum quantization condition $mvr = n\hbar$ reads

$$\sqrt{ma^2kV_0}\left(\frac{r}{a}\right)^{\frac{k+2}{2}} = n\hbar$$

We may use the result of this and the previous equation to calculate

$$E = \frac{1}{2}mv^2 + V_0\left(\frac{r}{a}\right)^k = (\frac{1}{2}k+1)V_0\left(\frac{r}{a}\right)^k = (\frac{1}{2}k+1)V_0\left[\frac{n^2\hbar^2}{ma^2kV_0}\right]^{\frac{k}{k+2}}$$

In the limit of $k \gg 1$, we get

$$E \to \frac{1}{2}(kV_0)^{\frac{2}{k+2}}\left[\frac{\hbar^2}{ma^2}\right]^{\frac{k}{k+2}}(n^2)^{\frac{k}{k+2}} \to \frac{\hbar^2}{2ma^2}n^2$$

Note that V_0 drops out of the result. This makes sense if one looks at a picture of the potential in the limit of large k. For $r < a$ the potential is effectively zero. For $r > a$ it is effectively infinite, simulating a box with infinite walls. The presence of V_0 is there to provide something with the dimensions of an energy. In the limit of the infinite box *with the quantum condition* there is no physical meaning to V_0 and the energy scale is provided by $\hbar^2/2ma^2$.

16. The condition $L = n\hbar$ implies that

$$E = \frac{n^2\hbar^2}{2I}$$

In a transition from n_1 to n_2 the Bohr rule implies that the frequency of the radiation is given

$$\nu_{12} = \frac{E_1 - E_2}{h} = \frac{\hbar^2}{2Ih}(n_1^2 - n_2^2) = \frac{\hbar}{4\pi I}(n_1^2 - n_2^2)$$

Let $n_1 = n_2 + \Delta n$. Then in the limit of large n we have $(n_1^2 - n_2^2) \to 2n_2\Delta n$, so that

$$\nu_{12} \to \frac{1}{2\pi}\frac{\hbar n_2}{I}\Delta n = \frac{1}{2\pi}\frac{L}{I}\Delta n$$

Classically the radiation frequency is the frequency of rotation which is $\omega = L/I$, i.e.

$$\nu_{cl} = \frac{\omega}{2\pi}\frac{L}{I}$$

We see that this is equal to ν_{12} when $\Delta n = 1$.

17. The energy gap between low-lying levels of rotational spectra is of the order of $\hbar^2 / I = (1/2\pi)h\hbar / MR^2$, where M is the reduced mass of the two nuclei, and R is their separation. (Equivalently we can take 2 x $m(R/2)^2 = MR^2$). Thus

$$h\nu = \frac{hc}{\lambda} = \frac{1}{2\pi}h\frac{\hbar}{MR^2}$$

This implies that

$$R = \sqrt{\frac{\hbar\lambda}{2\pi Mc}} = \sqrt{\frac{\hbar\lambda}{\pi mc}} = \sqrt{\frac{(1.05\times10^{-34}\,J.s)(10^{-3}\,m)}{\pi(1.67\times10^{-27}kg)(3\times10^8\,m/s)}} = 26\,nm$$

CHAPTER 2

1. We have

$$\psi(x)=\int_{-\infty}^{\infty} dkA(k)e^{ikx} = \int_{-\infty}^{\infty} dk\frac{N}{k^2+\alpha^2}e^{ikx} = \int_{-\infty}^{\infty} dk\frac{N}{k^2+\alpha^2}\cos kx$$

because only the even part of $e^{ikx} = \cos kx + i \sin kx$ contributes to the integral. The integral can be looked up. It yields

$$\psi(x)=N\frac{\pi}{\alpha}e^{-\alpha|x|}$$

so that

$$|\psi(x)|^2=\frac{N^2\pi^2}{\alpha^2}e^{-2\alpha|x|}$$

If we look at $|A(k)^2$ we see that this function drops to 1/4 of its peak value at $k =\pm \alpha$.. We may therefore estimate the width to be $\Delta k = 2\alpha$. The square of the wave function drops to about 1/3 of its value when
$x =\pm 1/2\alpha$. This choice then gives us $\Delta k\, \Delta x = 1$. Somewhat different choices will give slightly different numbers, but in all cases the product of the widths is independent of α.

2. the definition of the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{2\pi d\nu}{2\pi d(1/\lambda)} = \frac{d\nu}{d(1/\lambda)} = -\lambda^2\frac{d\nu}{d\lambda}$$

The relation between wavelength and frequency may be rewritten in the form

$$\nu^2 - \nu_0^2 = \frac{c^2}{\lambda^2}$$

so that

$$-\lambda^2\frac{d\nu}{d\lambda} = \frac{c^2}{\nu\lambda} = c\sqrt{1-(\nu_0/\nu)^2}$$

3. We may use the formula for $\boldsymbol{v_g}$ derived above for

$$\nu = \sqrt{\frac{2\pi T}{\rho}}\lambda^{-3/2}$$

to calculate

$$v_g = -\lambda^2 \frac{d\nu}{d\lambda} = \frac{3}{2}\sqrt{\frac{2\pi T}{\rho\lambda}}$$

4. For deep gravity waves,

$$\nu = \sqrt{g/2\pi}\,\lambda^{-1/2}$$

from which we get, in exactly the same way $v_g = \frac{1}{2}\sqrt{\frac{\lambda g}{2\pi}}$.

5. With $\omega = \hbar k^2/2m$, $\beta = \hbar/m$ and with the original width of the packet $w(0) = \sqrt{2}\alpha$, we have

$$\frac{w(t)}{w(0)} = \sqrt{1+\frac{\beta^2 t^2}{2\alpha^2}} = \sqrt{1+\frac{\hbar^2 t^2}{2m^2\alpha^2}} = \sqrt{1+\frac{2\hbar^2 t^2}{m^2 w^4(0)}}$$

(a) With t = 1 s, $m = 0.9 \times 10^{-30}$ kg and $w(0) = 10^{-6}$ m, the calculation yields $w(1) = 1.7 \times 10^2$ m

With $w(0) = 10^{-10}$ m, the calculation yields $w(1) = 1.7 \times 10^6$ m.

These are very large numbers. We can understand them by noting that the characteristic velocity associated with a particle spread over a range Δx is $v = \hbar/m\Delta x$ and here m is very small.

(b) For an object with mass 10^{-3} kg and $w(0) = 10^{-2}$ m, we get

$$\frac{2\hbar^2 t^2}{m^2 w^4(0)} = \frac{2(1.05\times 10^{-34}\,J.s)^2 t^2}{(10^{-3}\,kg)^2 \times (10^{-2} m)^4} = 2.2\times 10^{-54}$$

for t = 1. This is a totally negligible quantity so that $w(t) = w(0)$.

6. For the 13.6 eV electron v/c = 1/137, so we may use the nonrelativistic expression for the kinetic energy. We may therefore use the same formula as in problem **5**, that is

$$\frac{w(t)}{w(0)} = \sqrt{1+\frac{\beta^2 t^2}{2\alpha^2}} = \sqrt{1+\frac{\hbar^2 t^2}{2m^2\alpha^2}} = \sqrt{1+\frac{2\hbar^2 t^2}{m^2 w^4(0)}}$$

We caclulate t for a distance of 10^4 km = 10^7 m, with speed (3×10^8m/137) to be 4.6 s. We are given that $w(0) = 10^{-3}$ m. In that case

$$w(t) = (10^{-3} m)\sqrt{1+\frac{2(1.05\times 10^{-34}\,J.s)^2(4.6s)^2}{(0.9\times 10^{-30} kg)^2(10^{-3} m)^4}} = 7.5\times 10^{-2} m$$

For a 100 MeV electron $E = pc$ to a very good approximation. This means that $\beta = 0$ and therefore the packet does not spread.

7. For any massless particle $E = pc$ so that β= 0 and there is no spreading.

8. We have

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx A e^{-\mu|x|} e^{-ipx/\hbar} = \frac{A}{\sqrt{2\pi\hbar}} \left\{ \int_{-\infty}^{0} dx e^{(\mu - ik)x} + \int_{0}^{\infty} dx e^{-(\mu+ik)x} \right\}$$
$$= \frac{A}{\sqrt{2\pi\hbar}} \left\{ \frac{1}{\mu - ik} + \frac{1}{\mu + ik} \right\} = \frac{A}{\sqrt{2\pi\hbar}} \frac{2\mu}{\mu^2 + k^2}$$

where $k = p/\hbar$.

9. We want

$$\int_{-\infty}^{\infty} dx A^2 e^{-2\mu|x|} = A^2 \left\{ \int_{-\infty}^{0} dx e^{2\mu x} + \int_{0}^{\infty} dx e^{-2\mu x} \right\} = A^2 \frac{1}{\mu} = 1$$

so that

$$A = \sqrt{\mu}$$

10. Done in text.

11. Consider the Schrodinger equation with $V(x)$ complex. We now have

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{i}{\hbar} V(x) \psi(x,t)$$

and

$$\frac{\partial \psi^*(x,t)}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \psi^*(x,t)}{\partial x^2} + \frac{i}{\hbar} V^*(x) \psi(x,t)$$

Now

$$\frac{\partial}{\partial t}(\psi^* \psi) = \frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t}$$
$$= \left(-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V^*(x) \psi^* \right) \psi + \psi^* \left(\frac{i\hbar}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{i}{\hbar} V(x) \psi(x,t) \right)$$
$$= -\frac{i\hbar}{2m} \left(\frac{\partial^2 \psi^*}{\partial x^2} \psi - \psi^* \frac{\partial^2 \psi(x,t)}{\partial x^2} \right) + \frac{i}{\hbar} (V^* - V) \psi^* \psi$$

$$= -\frac{i\hbar}{2m} \frac{\partial}{\partial x} \left\{ \frac{\partial \psi^*}{\partial x} \psi - \psi^* \frac{\partial \psi}{\partial x} \right\} + \frac{2\,\mathrm{Im} V(x)}{\hbar} \psi^* \psi$$

Consequently

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\,|\psi(x,t)|^2 = \frac{2}{\hbar}\int_{-\infty}^{\infty} dx(\mathrm{Im}V(x))\,|\psi(x,t)|^2$$

We require that the left hand side of this equation is negative. This does not tell us much about Im*V(x)*
except that it cannot be positive everywhere. If it has a fixed sign, it must be *negative.*

12. The problem just involves simple arithmetic. The class average

$$\langle g\rangle = \sum_g g n_g = 38.5$$

$$(\Delta g)^2 = \langle g^2\rangle - \langle g\rangle^2 = \sum_g g^2 n_g - (38.5)^2 = \text{ 1570.8-1482.3= 88.6}$$

The table below is a result of the numerical calculations for this system

g	n_g	$(g - \langle g\rangle)^2/(\Delta g)^2 = \lambda$	$e^{-\lambda}$	$Ce^{-\lambda}$
60	1	5.22	0.0054	0.097
55	2	3.07	0.0463	0.833
50	7	1.49	0.2247	4.04
45	9	0.48	0.621	11.16
40	16	0.025	0.975	17.53
35	13	0.138	0.871	15.66
30	3	0.816	0.442	7.96
25	6	2.058	0.128	2.30
20	2	3.864	0.021	0.38
15	0	6.235	0.002	0.036
10	1	9.70	0.0001	0.002
5	0	12.97	"0"	"0"

15. We want

$$1 = 4N^2\int_{-\infty}^{\infty} dx\frac{\sin^2 kx}{x^2} = 4N^2 k\int_{-\infty}^{\infty} dt\frac{\sin^2 t}{t^2} = 4\pi N^2 k$$

so that $N = \sqrt{\dfrac{1}{4\pi k}}$

16. We have

$$\langle x^n \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx x^n e^{-\alpha x^2}$$

Note that this integral vanishes for n an odd integer, because the rest of the integrand is even.

For $n = 2m$, an even integer, we have

$$\langle x^{2m} \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} = \left(\frac{\alpha}{\pi}\right)^{1/2} \left(-\frac{d}{d\alpha}\right)^m \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \left(\frac{\alpha}{\pi}\right)^{1/2} \left(-\frac{d}{d\alpha}\right)^m \left(\frac{\pi}{\alpha}\right)^{1/2}$$

For $n = 1$ as well as $n = 17$ this is zero, while for $n = 2$, that is, $m = 1$, this is $\frac{1}{2\alpha}$.

17. $$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

The integral is easily evaluated by rewriting the exponent in the form

$$-\frac{\alpha}{2} x^2 - ix\frac{p}{\hbar} = -\frac{\alpha}{2}\left(x + \frac{ip}{\hbar\alpha}\right)^2 - \frac{p^2}{2\hbar^2\alpha}$$

A shift in the variable x allows us to state the value of the integral as and we end up with

$$\phi(p) = \frac{1}{\sqrt{\pi\hbar}} \left(\frac{\pi}{\alpha}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}$$

We have, for n even, i.e. $n = 2m$,

$$\langle p^{2m} \rangle = \frac{1}{\pi\hbar} \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dp p^{2m} e^{-p^2/\alpha\hbar^2} =$$

$$= \frac{1}{\pi\hbar} \left(\frac{\pi}{\alpha}\right)^{1/2} \left(-\frac{d}{d\beta}\right)^m \left(\frac{\pi}{\beta}\right)^{1/2}$$

where at the end we set $\beta = \frac{1}{\alpha\hbar^2}$. For odd powers the integral vanishes.

18. Specifically for $m = 1$ we have   We have

$$(\Delta x)^2 = \langle x^2 \rangle = \frac{1}{2\alpha}$$

$$(\Delta p)^2 = \langle p^2 \rangle = \frac{\alpha\hbar^2}{2}$$

so that $\Delta p \Delta x = \frac{\hbar}{2}$. This is, in fact, the smallest value possible for the product of the dispersions.

22. We have

$$\int_{-\infty}^{\infty} dx \psi^*(x) x \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi^*(x) x \int_{-\infty}^{\infty} dp \phi(p) e^{ipx/\hbar}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \psi^*(x) \int_{-\infty}^{\infty} dp \phi(p) \frac{\hbar}{i} \frac{\partial}{\partial p} e^{ipx/\hbar} = \int_{-\infty}^{\infty} dp \phi^*(p) i\hbar \frac{\partial \phi(p)}{\partial p}$$

In working this out we have shamelessly interchanged orders of integration. The justification of this is that the wave functions are expected to go to zero at infinity faster than any power of x, and this is also true of the momentum space wave functions, in their dependence on p.

CHAPTER 3.

1. The linear operators are (a), (b), (f)

2.We have

$$\int_{-\infty}^{x} dx' x' \psi(x') = \lambda \psi(x)$$

To solve this, we differentiate both sides with respect to *x*, and thus get

$$\lambda \frac{d\psi(x)}{dx} = x\psi(x)$$

A solution of this is obtained by writing $d\psi/\psi = (1/\lambda)xdx$ from which we can immediately state that

$$\psi(x) = Ce^{\lambda x^2/2}$$

The existence of the integral that defines $O_6\psi(x)$ requires that $\lambda < 0$.

3, (a)

$$\begin{aligned}
&O_2 O_6 \psi(x) - O_6 O_2 \psi(x) \\
&= x\frac{d}{dx}\int_{-\infty}^{x} dx' x' \psi(x') - \int_{-\infty}^{x} dx' x'^2 \frac{d\psi(x')}{dx'} \\
&= x^2\psi(x) - \int_{-\infty}^{x} dx' \frac{d}{dx'}\left(x'^2\,\psi(x')\right) + 2\int_{-\infty}^{x} dx' x' \psi(x') \\
&= 2O_6\psi(x)
\end{aligned}$$

Since this is true for every $\psi(x)$ that vanishes rapidly enough at infinity, we conclude that

$$[O_2 , O_6] = 2O_6$$

(b)

$$\begin{aligned}
&O_1 O_2 \psi(x) - O_2 O_1 \psi(x) \\
&= O_1\left(x\frac{d\psi}{dx}\right) - O_2\left(x^3\psi\right) = x^4\frac{d\psi}{dx} - x\frac{d}{dx}\left(x^3\psi\right) \\
&= -3x^3\psi(x) = -3O_1\psi(x)
\end{aligned}$$

so that

$$[O_1, O_2] = -3O_1$$

4. We need to calculate

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a dx x^2 \sin^2 \frac{n\pi x}{a}$$

With $\pi x/a = \mathrm{u}$ we have

$$\langle x^2 \rangle = \frac{2}{a}\frac{a^3}{\pi^3}\int_0^\pi du u^2 \sin^2 nu = \frac{a^2}{\pi^3}\int_0^\pi du u^2 (1-\cos 2nu)$$

The first integral is simple. For the second integral we use the fact that

$$\int_0^\pi du u^2 \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \int_0^\pi du \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \frac{\sin\alpha\pi}{\alpha}$$

At the end we set $\alpha = n\pi$. A little algebra leads to

$$\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2}$$

For large n we therefore get $\Delta x = \frac{a}{\sqrt{3}}$. Since $\langle p^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}$, it follows that $\Delta p = \frac{\hbar \pi n}{a}$, so that

$$\Delta p \Delta x \approx \frac{n\pi\hbar}{\sqrt{3}}$$

The product of the uncertainties thus grows as n increases.

5. With $E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2$ we can calculate

$$E_2 - E_1 = 3\frac{(1.05 \times 10^{-34} J.s)^2}{2(0.9 \times 10^{-30} kg)(10^{-9} m)^2}\frac{1}{(1.6 \times 10^{-19} J/eV)} = 0.115 eV$$

We have $\Delta E = \frac{hc}{\lambda}$ so that $\lambda = \frac{2\pi\hbar c}{\Delta E} = \frac{2\pi(2.6 \times 10^{-7} ev.m)}{0.115 eV} = 1.42 \times 10^{-5} m$

where we have converted $\hbar c$ from J.m units to eV.m units.

6. (a) Here we write

$$n^2 = \frac{2ma^2E}{\hbar^2\pi^2} = \frac{2(0.9\times10^{-30}kg)(2\times10^{-2}m)^2(1.5eV)(1.6\times10^{-19}J/eV)}{(1.05\times10^{-34}J.s)^2\pi^2} = 1.59\times10^{15}$$

so that $n = 4 \times 10^7$.

(b) We have

$$\Delta E = \frac{\hbar^2\pi^2}{2ma^2}2n\Delta n = \frac{(1.05\times10^{-34}J.s)^2\pi^2}{2(0.9\times10^{-30}kg)(2\times10^{-2}m)^2}2(4\times10^7) = 1.2\times10^{-26}J$$
$$= 7.6\times10^{-8}eV$$

7. The longest wavelength corresponds to the lowest frequency. Since ΔE is proportional to $(n + 1)^2 - n^2 = 2n + 1$, the lowest value corresponds to $n = 1$ (a state with $n = 0$ does not exist). We therefore have

$$h\frac{c}{\lambda} = 3\frac{\hbar^2\pi^2}{2ma^2}$$

If we assume that we are dealing with electrons of mass $m = 0.9 \times 10^{-30}$ kg, then

$$a^2 = \frac{3\hbar\pi\lambda}{4mc} = \frac{3\pi(1.05\times10^{-34}J.s)(4.5\times10^{-7}m)}{4(0.9\times10^{-30}kg)(3\times10^8 m/s)} = 4.1\times10^{-19}m^2$$

so that $a = 6.4 \times 10^{-10}$ m.

8. The solutions for a box of width a have energy eigenvalues $E_n = \frac{\hbar^2\pi^2n^2}{2ma^2}$ with $n = 1,2,3,\ldots$The odd integer solutions correspond to solutions even under $x \to -x$, while the even integer solutions correspond to solutions that are odd under reflection. These solutions vanish at $x = 0$, and it is these solutions that will satisfy the boundary conditions for the "half-well" under consideration. Thus the energy eigenvalues are given by E_n above *with n even*.

9. The general solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n u_n(x) e^{-iE_n t/\hbar}$$

with the C_n defined by

$$C_n = \int_{-a/2}^{a/2} dx u_n^*(x)\psi(x,0)$$

(a) It is clear that the wave function does not remain localized on the l.h.s. of the box at later times, since the special phase relationship that allows for a total interference for $x > 0$ no longer persists for $t \neq 0$.

(b) With our wave function we have $C_n = \sqrt{\frac{2}{a}} \int_{-q/2}^{0} dx u_n(x)$. We may work this out by using the solution of the box extending from $x = 0$ to $x = a$, since the shift has no physical consequences. We therefore have

$$C_n = \sqrt{\frac{2}{a}} \int_0^{a/2} dx \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} = \frac{2}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^{a/2} = \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$$

Therefore $P_1 = |C_1|^2 = \frac{4}{\pi^2}$ and $P_2 = |C_2|^2 = \frac{1}{\pi^2}|(1-(-1))|^2 = \frac{4}{\pi^2}$

10. (a) We use the solution of the above problem to get

$$P_n = |C_n|^2 = \frac{4}{n^2\pi^2} f_n$$

where $f_n = 1$ for n = odd integer; $f_n = 0$ for $n = 4,8,12,\ldots$ and $f_n = 4$ for $n = 2,6,10,\ldots$

(b) We have

$$\sum_{n=1}^{\infty} P_n = \frac{4}{\pi^2}\sum_{odd}\frac{1}{n^2} + \frac{4}{\pi^2}\sum_{n=2,6,10,,,}\frac{4}{n^2} = \frac{8}{\pi^2}\sum_{odd}\frac{1}{n^2} = 1$$

Note. There is a typo in the statement of the problem. The sum should be restricted to *odd* integers.

11. We work this out by making use of an identity. The hint tells us that

$$(\sin x)^5 = \left(\frac{1}{2i}\right)^5 (e^{ix} - e^{-ix})^5 = \frac{1}{16}\frac{1}{2i}(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix})$$
$$= \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$$

Thus

$$\psi(x,0) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x) - 5u_3(x) + 10u_1(x)\right)$$

(a) It follows that

$$\psi(x,t) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x)e^{-iE_5t/\hbar} - 5u_3(x)e^{-iE_3t/\hbar} + 10u_1(x)e^{-iE_1t/\hbar}\right)$$

(b) We can calculate A by noting that $\int_0^a dx\,|\psi(x,0)|^2 = 1$. This however is equivalent to the statement that the sum of the probabilities of finding *any* energy eigenvalue adds up to 1. Now we have

$$P_5 = \frac{a}{2}A^2\frac{1}{256}; P_3 = \frac{a}{2}A^2\frac{25}{256}; P_1 = \frac{a}{2}A^2\frac{100}{256}$$

so that

$$A^2 = \frac{256}{63a}$$

The probability of finding the state with energy E_3 is 25/126.

12. The initial wave function vanishes for $x \leq -a$ and for $x \geq a$. In the region in between it is proportional to $\cos\frac{\pi x}{2a}$, since this is the first nodeless trigonometric function that vanishes at $x = \pm a$. The normalization constant is obtained by requiring that

$$1 = N^2\int_{-a}^{a} dx\cos^2\frac{\pi x}{2a} = N^2\left(\frac{2a}{\pi}\right)\int_{-\pi/2}^{\pi/2} du\cos^2 u = N^2 a$$

so that $N = \sqrt{\frac{1}{a}}$. We next expand this in eigenstates of the infinite box potential with boundaries at $x = \pm b$. We write

$$\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a} = \sum_{n=1}^{\infty} C_n u_n(x;b)$$

so that

$$C_n = \int_{-b}^{b} dx u_n(x;b)\psi(x) = \int_{-a}^{a} dx u_n(x;b)\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a}$$

In particular, after a little algebra, using cosu cosv=(1/2)[cos(u-v)+cos(u+v)], we get

$$C_1 = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \cos\frac{\pi x}{2b}\cos\frac{\pi x}{2a} = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \frac{1}{2}\left[\cos\frac{\pi x(b-a)}{2ab} + \cos\frac{\pi x(b+a)}{2ab}\right]$$
$$= \frac{4b\sqrt{ab}}{\pi(b^2-a^2)}\cos\frac{\pi a}{2b}$$

so that

$$P_1 = |C_1|^2 = \frac{16ab^3}{\pi^2(b^2-a^2)^2}\cos^2\frac{\pi a}{2b}$$

The calculation of C_2 is trivial. The reason is that while $\psi(x)$ is an *even* function of x, $u_2(x)$ is an *odd* function of x, and the integral over an interval symmetric about $x = 0$ is zero. Hence P_2 will
be zero.

13. We first calculate

$$\phi(p) = \int_0^a dx\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} = \frac{1}{i}\sqrt{\frac{1}{4\pi\hbar a}}\left(\int_0^a dx e^{ix(n\pi/a+p/\hbar)} - (n \leftrightarrow -n)\right)$$
$$= \sqrt{\frac{1}{4\pi\hbar a}}\left(\frac{e^{iap/\hbar}(-1)^n - 1}{p/\hbar - n\pi/a} - \frac{e^{iap/\hbar}(-1)^n - 1}{p/\hbar + n\pi/a}\right)$$
$$= \sqrt{\frac{1}{4\pi\hbar a}}\frac{2n\pi/a}{(n\pi/a)^2 - (p/\hbar)^2}\left\{(-1)^n\cos pa/\hbar - 1 + i(-1)^n\sin pa/\hbar\right\}$$

From this we get

$$P(p) = |\phi(p)|^2 = \frac{2n^2\pi}{a^3\hbar}\frac{1-(-1)^n\cos pa/\hbar}{\left[(n\pi/a)^2 - (p/\hbar)^2\right]^2}$$

The function $P(p)$ does not go to infinity at $p = n\pi\hbar/a$, but if definitely peaks there. If we write $p/\hbar = n\pi/a + \varepsilon$, then the numerator becomes $1-\cos a\varepsilon \approx a^2\varepsilon^2/2$ and the denominator becomes $(2n\pi\varepsilon/a)^2$, so that at the peak $P\left(\frac{n\pi\hbar}{a}\right) = a/4\pi\hbar$. The fact that the peaking occurs at

$$\frac{p^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$$

suggests agreement with the correspondence principle, since the kinetic energy of the particle is, as the r.h.s. of this equation shows, just the energy of a particle in the infinite box of width a. To confirm this, we need to show that the distribution is strongly peaked for large n. We do this by looking at the numerator, which vanishes when $a\varepsilon = \pi/2$, that is, when $p/\hbar = n\pi/a + \pi/2a = (n+1/2)\pi/a$. This implies that the width of the distribution is $\Delta p = \pi\hbar / 2a$. Since the x-space wave function is localized to $0 \le x \le a$ we only know that $\Delta x = a$. The result $\Delta p \Delta x \approx (\pi/2)\hbar$ is consistent with the uncertainty principle.

14. We calculate

$$\begin{aligned}\phi(p) &= \int_{-\infty}^{\infty} dx \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \\ &= \left(\frac{\alpha}{\pi}\right)^{1/4} \left(\frac{1}{2\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} dx e^{-\alpha(x-ip/\alpha\hbar)^2} e^{-p^2/2\alpha\hbar^2} \\ &= \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}\end{aligned}$$

From this we find that the probability the momentum is in the range $(p, p + dp)$ is

$$|\phi(p)|^2 \, dp = \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} e^{-p^2/\alpha\hbar^2}$$

To get the expectation value of the energy we need to calculate

$$\begin{aligned}\langle \frac{p^2}{2m} \rangle &= \frac{1}{2m}\left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} dp p^2 e^{-p^2/\alpha\hbar^2} \\ &= \frac{1}{2m}\left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} \frac{\sqrt{\pi}}{2} (\alpha\hbar^2)^{3/2} = \frac{\alpha\hbar^2}{2m}\end{aligned}$$

An estimate on the basis of the uncertainty principle would use the fact that the "width" of the packet is $1/\sqrt{\alpha}$. From this we estimate $\Delta p \approx \hbar / \Delta x = \hbar\sqrt{\alpha}$, so that

$$E \approx \frac{(\Delta p)^2}{2m} = \frac{\alpha\hbar^2}{2m}$$

The *exact* agreement is fortuitous, since both the definition of the width and the numerical statement of the uncertainty relation are somewhat elastic.

15. We have

$$j(x) = \frac{\hbar}{2im}\left(\psi^*(x)\frac{d\psi(x)}{dx} - \frac{d\psi^*(x)}{dx}\psi(x)\right)$$

$$= \frac{\hbar}{2im}\left[(A^* e^{-ikx} + B^* e^{ikx})(ikAe^{ikx} - ikBe^{-ikx}) - c.c)\right]$$

$$= \frac{\hbar}{2im}[ik|A|^2 - ik|B|^2 + ikAB^* e^{2ikx} - ikA^* Be^{-2ikx}$$

$$-(-ik)|A|^2 - (ik)|B|^2 - (-ik)A^* Be^{-2ikx} - ikAB^* e^{2ikx}]$$

$$= \frac{\hbar k}{m}[|A|^2 - |B|^2]$$

This is a sum of a flux to the right associated with $A\, e^{ikx}$ and a flux to the left associated with Be^{-ikx}..

16. Here

$$j(x) = \frac{\hbar}{2im}\left[u(x)e^{-ikx}(iku(x)e^{ikx} + \frac{du(x)}{dx}e^{ikx}) - c.c\right]$$

$$= \frac{\hbar}{2im}[(iku^2(x) + u(x)\frac{du(x)}{dx}) - c.c] = \frac{\hbar k}{m}u^2(x)$$

(c) Under the reflection $x \rightarrow -x$ both x and $p = -i\hbar\frac{\partial}{\partial x}$ change sign, and since the function consists of an odd power of x and/or p, it is an odd function of x. Now the eigenfunctions for a box symmetric about the x axis have a definite parity. So that $u_n(-x) = \pm u_n(x)$. This implies that the integrand is *antisymmetric* under $x \rightarrow -x$. Since the integral is over an interval symmetric under this exchange, it is zero.

(d) We need to prove that

$$\int_{-\infty}^{\infty} dx (P\psi(x))^* \psi(x) = \int_{-\infty}^{\infty} dx \psi(x)^* P\psi(x)$$

The left hand side is equal to

$$\int_{-\infty}^{\infty} dx \psi^*(-x)\psi(x) = \int_{-\infty}^{\infty} dy\, \psi^*(y)\psi(-y)$$

with a change of variables $x \rightarrow -y$, and this is equal to the right hand side.

The eigenfunctions of P with eigenvalue +1 are functions for which $u(x) = u(-x)$, while those with eigenvalue –1 satisfy $v(x) = -v(-x)$. Now the scalar product is

$$\int_{-\infty}^{\infty} dx u^*(x)v(x) = \int_{-\infty}^{\infty} dy u^*(-x)v(-x) = -\int_{-\infty}^{\infty} dx u^*(x)v(x)$$

so that

$$\int_{-\infty}^{\infty} dx u^*(x)v(x) = 0$$

(e) A simple sketch of $\psi(x)$ shows that it is a function symmetric about $x = a/2$. This means that the integral $\int_0^a dx \psi(x) u_n(x)$ will vanish for the $u_n(x)$ which are *odd* under the reflection about this axis. This means that the integral vanishes for $n = 2,4,6,\ldots$

CHAPTER 4.

1. The solution to the left side of the potential region is $\psi(x) = Ae^{ikx} + Be^{-ikx}$. As shown in Problem 3-15, this corresponds to a flux

$$j(x) = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$

The solution on the right side of the potential is $\psi(x) = Ce^{ikx} + De^{-ikxx}$, and as above, the flux is

$$j(x) = \frac{\hbar k}{m}\left(|C|^2 - |D|^2\right)$$

Both fluxes are independent of *x*. Flux conservation implies that the two are equal, and this leads to the relationship

$$|A|^2 + |D|^2 = |B|^2 + |C|^2$$

If we now insert

$$C = S_{11}A + S_{12}D$$
$$B = S_{21}A + S_{22}D$$

into the above relationship we get

$$|A|^2 + |D|^2 = (S_{21}A + S_{22}D)(S_{21}^*A^* + S_{22}^*D^*) + (S_{11}A + S_{12}D)(S_{11}^*A^* + S_{12}^*D^*)$$

Identifying the coefficients of $|A|^2$ and $|D|^2$, and setting the coefficient of AD^* equal to zero yields

$$|S_{21}|^2 + |S_{11}|^2 = 1$$
$$|S_{22}|^2 + |S_{12}|^2 = 1$$
$$S_{12}S_{22}^* + S_{11}S_{12}^* = 0$$

Consider now the matrix

$$S^{tr} = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}$$

The unitarity of this matrix implies that

$$\begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}\begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

that is,

$$|S_{11}|^2 + |S_{21}|^2 = |S_{12}|^2 + |S_{22}|^2 = 1$$

$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0$$

These are just the conditions obtained above. They imply that the matrix S^{tr} is unitary, and therefore the matrix S is unitary.

2. We have solve the problem of finding R and T for this potential well in the text.We take $V_0 < 0$. We dealt with wave function of the form

$$e^{ikx} + Re^{-ikx} \qquad x < -a$$

$$Te^{ikx} \qquad x > a$$

In the notation of Problem 4-1, we have found that if $A = 1$ and $D = 0$, then $C = S_{11} = T$ and $B = S_{21} = R$.. To find the other elements of the S matrix we need to consider the same problem with $A = 0$ and $D = 1$. This can be solved explicitly by matching wave functions at the boundaries of the potential hole, but it is possible to take the solution that we have and reflect the "experiment" by the interchange $x \rightarrow -x$. We then find that $S_{12} = R$ and $S_{22} = T$. We can easily check that

$$|S_{11}|^2 + |S_{21}|^2 = |S_{12}|^2 + |S_{22}|^2 = |R|^2 + |T|^2 = 1$$

Also

$$S_{11}S_{12}^* + S_{21}S_{22}^* = TR^* + RT^* = 2\,\mathrm{Re}(TR^*)$$

If we now look at the solutions for T and R in the text we see that the product of T and R^* is of the form (-i) x (real number), so that its real part is zero. This confirms that the S matrix here is unitary.

3. Consider the wave functions on the left and on the right to have the forms

$$\psi_L(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_R(x) = Ce^{ikx} + De^{-ikx}$$

Now, let us make the change $k \rightarrow -k$ *and* complex conjugate everything. Now the two wave functions read

$$\psi_L(x)' = A^* e^{ikx} + B^* e^{-ikx}$$
$$\psi_R(x)' = C^* e^{ikx} + D^* e^{-ikx}$$

Now complex conjugation and the transformation $k \rightarrow -k$ changes the original relations to

$$C^* = S_{11}^*(-k)A^* + S_{12}^*(-k)D^*$$
$$B^* = S_{21}^*(-k)A^* + S_{22}^*(-k)D^*$$

On the other hand, we are now relating outgoing amplitudes C^*, B^* to ingoing amplitude A^*, D^*, so that the relations of problem 1 read

$$C^* = S_{11}(k)A^* + S_{12}(k)D^*$$
$$B^* = S_{21}(k)A^* + S_{22}(k)D^*$$

This shows that $S_{11}(k) = S_{11}^*(-k);\ \ S_{22}(k) = S_{22}^*(-k);\ \ S_{12}(k) = S_{21}^*(-k)$. These result may be written in the *matrix* form $\mathbf{S}(k) = \mathbf{S}^+(-k)$.

4. (a) With the given flux, the wave coming in from $x = -\infty$, has the form e^{ikx}, with unit amplitude. We now write the solutions in the various regions

$x < b$ $\quad e^{ikx} + Re^{-ikx} \quad k^2 = 2mE/\hbar^2$

$-b < x < -a$ $\quad Ae^{\kappa x} + Be^{-\kappa x} \quad \kappa^2 = 2m(V_0 - E)/\hbar^2$

$-a < x < c$ $\quad Ce^{ikx} + De^{-ikx}$

$c < x < d$ $\quad Me^{iqx} + Ne^{-iqx} \quad q^2 = 2m(E + V_1)/\hbar^2$

$d < x$ $\quad Te^{ikx}$

(b) We now have

$x < 0$ $\quad u(x) = 0$

$0 < x < a$ $\quad A\sin kx \quad k^2 = 2mE/\hbar^2$

$a < x < b$ $\quad Be^{\kappa x} + Ce^{-\kappa x} \quad \kappa^2 = 2m(V_0 - E)/\hbar^2$

$b < x$ $\quad e^{-ikx} + Re^{ikx}$

The fact that there is total reflection at $x = 0$ implies that $|R|^2 = 1$

5. The denominator in (4-   ) has the form

$$D = 2kq\cos 2qa - i(q^2 + k^2)\sin 2qa$$

With $k = i\kappa$ this becomes

$$D = i\left(2\kappa q\cos 2qa - (q^2 - \kappa^2)\sin 2qa\right)$$

The denominator vanishes when

$$\tan 2qa = \frac{2\tan qa}{1 - \tan^2 qa} = \frac{2q\kappa}{q^2 - \kappa^2}$$

This implies that

$$\tan qa = -\frac{q^2 - \kappa^2}{2\kappa q} \pm \sqrt{1 + \left(\frac{q^2 - \kappa^2}{2\kappa q}\right)^2} = -\frac{q^2 - \kappa^2}{2\kappa q} \pm \frac{q^2 + \kappa^2}{2\kappa q}$$

This condition is identical with (4-   ).

The argument why this is so, is the following: When $k = i\kappa$ the wave functio on the left has the form $e^{-\kappa x} + R(i\kappa)e^{\kappa x}$. The function $e^{-\kappa x}$ blows up as $x \to -\infty$ and the wave function only make sense if this term is overpowered by the other term, that is when $R(i\kappa) = \infty$. We leave it to the student to check that the numerators are the same at $k = i\kappa$.

6. The solution is $u(x) = Ae^{ikx} + Be^{-ikx} \quad x < b$

$\qquad = Ce^{ikx} + De^{-ikx} \quad x > b$

The continuity condition at $x = b$ leads to

$$Ae^{ikb} + Be^{-ikb} = Ce^{ikb} + De^{-ikb}$$

And the derivative condition is

$$(ikAe^{ikb} - ikBe^{-ikb}) - (ikCe^{ikb} - ikDe^{-ikb}) = (\lambda/a)(Ae^{ikb} + Be^{-ikb})$$

With the notation

$$Ae^{ikb} = \alpha\ ;\ Be^{-ikb} = \beta;\ Ce^{ikb} = \gamma;\ De^{-ikb} = \delta$$

These equations read

$$\alpha + \beta = \gamma + \delta$$

$$ik(\alpha - \beta + \gamma - \delta) = (\lambda/a)(\alpha + \beta)$$

We can use these equations to write (γ,β) in terms of (α,δ) as follows

$$\gamma = \frac{2ika}{2ika-\lambda}\alpha + \frac{\lambda}{2ika-\lambda}\delta$$
$$\beta = \frac{\lambda}{2ika-\lambda}\alpha + \frac{2ika}{2ika-\lambda}\delta$$

We can now rewrite these in terms of *A*,*B*,*C*,*D* and we get for the S matrix

$$S = \begin{pmatrix} \frac{2ika}{2ika-\lambda} & \frac{\lambda}{2ika-\lambda}e^{-2ikb} \\ \frac{\lambda}{2ika-\lambda}e^{2ikb} & \frac{2ika}{2ika-\lambda} \end{pmatrix}$$

Unitarity is easily established:

$$|S_{11}|^2 + |S_{12}|^2 = \frac{4k^2a^2}{4k^2a^2+\lambda^2} + \frac{\lambda^2}{4k^2a^2+\lambda^2} = 1$$
$$S_{11}S_{12}^* + S_{12}S_{22}^* = \left(\frac{2ika}{2ika-\lambda}\right)\left(\frac{\lambda}{-2ika-\lambda}e^{-2ikb}\right) + \left(\frac{\lambda}{2ika-\lambda}e^{-2ikb}\right)\left(\frac{-2ika}{-2ika-\lambda}\right) = 0$$

The matrix elements become infinite when $2ika = \lambda$. In terms of $\kappa = -ik$, this condition becomes $\kappa = -\lambda/2a = |\lambda|/2a$.

7. The exponent in $T = e^{-S}$ is

$$S = \frac{2}{\hbar}\int_A^B dx\sqrt{2m(V(x)-E)}$$
$$= \frac{2}{\hbar}\int_A^B dx\sqrt{(2m(\frac{m\omega^2}{2}(x^2 - \frac{x^3}{a})) - \frac{\hbar\omega}{2}}$$

where *A* and *B* are turning points, that is, the points at which the quantity under the square root sign vanishes.
We first simplify the expression by changing to dimensionless variables:

$$x = \sqrt{\hbar/m\omega}\,y;\;\; \eta = a/\sqrt{\hbar/m\omega} << 1$$

The integral becomes

$$2\int_{y_1}^{y_2} dy\sqrt{y^2-\eta y^3-1} \qquad \text{with} \quad \eta \ll 1$$

where now y_1 and y_2 are the turning points. A sketch of the potential shows that y_2 is very large. In that region, the –1 under the square root can be neglected, and to a good approximation $y_2 = 1/\eta$. The other turning point occurs for y not particularly large, so that we can neglect the middle term under the square root, and the value of y_1 is 1. Thus we need to estimate

$$\int_1^{1/\eta} dy\sqrt{y^2-\eta y^3-1}$$

The integrand has a maximum at $2y-3\eta y^2=0$, that is at $y = 2\eta/3$. We estimate the contribution from that point on by neglecting the –1 term in the integrand. We thus get

$$\int_{2/3\eta}^{1/\eta} dy y\sqrt{1-\eta y} = \frac{2}{\eta^2}\left[\frac{(1-\eta y)^{5/2}}{5}-\frac{(1-\eta y)^{3/2}}{3}\right]_{2/3\eta}^{1/\eta} = \frac{8\sqrt{3}}{135}\frac{1}{\eta^2}$$

To estimate the integral in the region $1 < y < 2/3\eta$ is more difficult. In any case, we get a lower limit on S by just keeping the above, so that

$$S > 0.21/\eta^2$$

The factor e^S must be multiplied by a characteristic time for the particle to move back and forth inside the potential with energy $\hbar\omega/2$ which is necessarily of order $1/\omega$. Thus the estimated time is *longer* than $\frac{const.}{\omega}e^{0.2/\eta^2}$.

8. The barrier factor is e^S where

$$S = \frac{2}{\hbar}\int_{R_0}^{b} dx\sqrt{\frac{\hbar^2 l(l+1)}{x^2}-2mE}$$

where b is given by the value of x at which the integrand vanishes, that is, with $2mE/\hbar^2=k^2$, $b=\sqrt{l(l+1)}/k$. We have, after some algebra

$$S = 2\sqrt{l(l+1)}\int_{R_0/b}^{1}\frac{du}{u}\sqrt{1-u^2}$$

$$= 2\sqrt{l(l+1)}\left[\ln\frac{1+\sqrt{1-(R_0/b)^2}}{R_0/b}-\sqrt{1-(R_0/b)^2}\right]$$

We now introduce the variable $f = (R_0/b) \approx kR_0/l$ for large l. Then

$$e^S e^S = \left[\frac{1+\sqrt{1-f^2}}{f}\right]^{2l} e^{-2l\sqrt{1-f^2}} \approx \left(\frac{e}{2}\right)^{-2l} f^{-2l}$$

for $f << 1$. This is to be multiplied by the time of traversal inside the box. The important factor is f^{2l}. It tells us that the lifetime is proportional to $(kR_0)^{-2l}$ so that it grows as a power of l for small k. Equivalently we can say that the probability of decay falls as $(kR_0)^{2l}$.

9. The argument fails because the electron is not localized inside the potential. In fact, for weak binding, the electron wave function extends over a region $R = 1/\alpha = \hbar\sqrt{2mE_B}$, which, for weak binding is much larger than a.

10. For a bound state, the solution for $x > a$ must be of the form $u(x) = Ae^{-\alpha x}$, where $\alpha = \sqrt{2mE_B}/\hbar$. Matching $\frac{1}{u}\frac{du}{dx}$ at $x = a$

yields $-\alpha = f(E_B)$. If $f(E)$ is a constant, then we immediately know α.. Even if $f(E)$ varies only slightly over the energy range that overlaps small positive E, we can determine the binding energy in terms of the reflection coefficient. For positive energies the wave function $u(x)$ for $x > a$ has the form $e^{-ikx} + R(k)e^{ikx}$, and matching yields

$$f(E) \approx -\alpha = -ik\frac{e^{-ika} - Re^{ika}}{e^{-ika} + Re^{ika}} = -ik\frac{1-Re^{2ika}}{1+Re^{2ika}}$$

so that

$$R = e^{-2ika}\frac{k+i\alpha}{k-i\alpha}$$

We see that $|R|^2 = 1$.

11. Since the well is symmetric about $x = 0$, we need only match wave functions at $x = b$ and a. We look at $E < 0$, so that we introduce and $\alpha^2 = 2m|E|/\hbar^2$ and $q^2 = 2m(V_0-|E|)/\hbar^2$. We now write down

Even solutions:

$$\begin{aligned} u(x) &= \cosh\alpha x & 0 < x < b \\ &= A\sin qx + B\cos qx & b < x < a \\ &= C e^{-\alpha x} & a < x \end{aligned}$$

Matching $\frac{1}{u(x)}\frac{du(x)}{dx}$ at $x = b$ and at $x = a$ leads to the equations

$$\alpha \tanh \alpha b = q\frac{A\cos qb - B\sin qb}{A\sin qb + B\cos qb}$$

$$-\alpha = q\frac{A\cos qa - Bsnqa}{A\sin qa + B\cos qa}$$

From the first equation we get

$$\frac{B}{A} = \frac{q\cos qb - \alpha \tanh \alpha b \sin qb}{q\sin qb + \alpha \tanh \alpha b \cos qb}$$

and from the second

$$\frac{B}{A} = \frac{q\cos qa + \alpha \sin qa}{q\sin qa - \alpha \cos qa}$$

Equating these, cross-multiplying, we get after a little algebra

$$q^2 \sin q(a-b) - \alpha \cos q(a-b) = \alpha \tanh \alpha b[\alpha \sin q(a-b) + q\cos q(a-b)]$$

from which it immediately follows that

$$\frac{\sin q(a-b)}{\cos q(a-b)} = \frac{\alpha q(\tanh \alpha b + 1)}{q^2 - \alpha^2 \tanh \alpha b}$$

Odd Solution

Here the only difference is that the form for $u(x)$ for $0 < x < b$ is $\sinh\alpha x$. The result of this is that we get the same expresion as above, with $\tanh\alpha b$ replaced by $\coth\alpha b$.

11. (a) The condition that there are at most two bound states is equivalent to stating that there is at most one *odd* bound state. The relevant figure is Fig. 4-8, and we ask for the condition that there be no intersection point with the tangent curve that starts up at $3\pi/2$. This means that

$$\frac{\sqrt{\lambda - y^2}}{y} = 0$$

for $y \le 3\pi/2$. This translates into $\lambda = y^2$ with $y < 3\pi/2$, i.e. $\lambda < 9\pi^2/4$.
(b) The condition that there be at most three bound states implies that there be at most two *even* bound states, and the relevant figure is 4-7. Here the conditon is that $y < 2\pi$ so that $\lambda < 4\pi^2$.

(c) We have $y = \pi$ so that the second *even* bound state have zero binding energy. This means that $\lambda = \pi^2$. What does this tell us about the first bound state? All we know is that y is a solution of Eq. (4-54) with $\lambda = \pi^2$. Eq.(4-54) can be rewritten as follows:

$$\tan^2 y = \frac{1-\cos^2 y}{\cos^2 y} = \frac{\lambda - y^2}{y^2} = \frac{1-(y^2/\lambda)}{(y^2/\lambda)}$$

so that the *even* condition is $\cos y = y/\sqrt{\lambda}$, and in the same way, the *odd* conditin is $\sin y = y/\sqrt{\lambda}$. Setting $\sqrt{\lambda} = \pi$ still leaves us with a transcendental equation. All we can say is that the binding energy f the even state will be larger than that of the odd one.

13.(*a*) As $b \to 0$, $\tan q(a-b) \to \tan qa$ and the r.h.s. reduces to α/q. Thus we get, for the even solution

$$\tan qa = \alpha/q$$

and, for the odd solution,

$$\tan qa = -q/\alpha.$$

These are just the single well conditions.
(b) This part is more complicated. We introduce notation $c = (a-b)$, which will be held fixed. We will also use the notation $z = \alpha b$. We will also use the subscript "1" for the even solutions, and "2" for the odd solutions. For b large,

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{1-e^{-2z}}{1+e^{-2z}} \approx 1 - 2e^{-2z}$$
$$\coth z \approx 1 = 2e^{-2z}$$

The eigenvalue condition for the even solution now reads

$$\tan q_1 c = \frac{q_1\alpha_1(1+1-2e^{-2z_1})}{q_1^2 - \alpha_1^2(1-2e^{-2z_1})} \approx \frac{2q_1\alpha_1}{q_1^2-\alpha_1^2}(1 - \frac{q_1^2+\alpha_1^2}{q_1^2-\alpha_1^2}e^{-2z_1})$$

The condition for the odd solution is obtained by just changing the sign of the e^{-2z} term, so that

$$\tan q_2 c = \frac{q_2\alpha_2(1+1+2e^{-2z_2})}{q_2^2\alpha_2^2(1+2e^{-2z_2})} \approx \frac{2q_2\alpha_2}{q_2^2-\alpha_2^2}(1 + \frac{q_2^2+\alpha_2^2}{q_2^2-\alpha_2^2}e^{-2z_2})$$

In both cases $q^2 + \alpha^2 = 2mV_0/\hbar^2$ is fixed. The two eigenvalue conditions only differ in the e^{-2z} terms, and the difference in the eigenvalues is therefore proportional to e^{-2z} , where z here is some mean value between $\alpha_1 b$ and $\alpha_2 b$.
This can be worked out in more detail, but this becomes an exercise in Taylor expansions with no new physical insights.

14. We write

$$\langle x\frac{dV(x)}{dx}\rangle = \int_{-\infty}^{\infty} dx\,\psi(x)x\frac{dV(x)}{dx}\psi(x)$$
$$= \int_{-\infty}^{\infty} dx\left[\frac{d}{dx}\left(\psi^2 xV\right) - 2\psi\frac{d\psi}{dx}xV - \psi^2 V\right]$$

The first term vanishes because ψ goes to zero rapidly. We next rewrite

$$-2\int_{-\infty}^{\infty} dx\frac{d\psi}{dx}xV\psi = -2\int_{-\infty}^{\infty} dx\frac{d\psi}{dx}x(E + \frac{\hbar^2}{2m}\frac{d^2}{dx^2})\psi$$
$$= -E\int_{-\infty}^{\infty} dxx\frac{d\psi^2}{dx} - \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dxx\frac{d}{dx}\left(\frac{d\psi}{dx}\right)^2$$

Now

$$\int_{-\infty}^{\infty} dxx\frac{d\psi^2}{dx} = \int_{-\infty}^{\infty} dx\frac{d}{dx}\left(x\psi^2\right) - \int_{-\infty}^{\infty} dx\psi^2$$

The first term vanishes, and the second term is unity. We do the same with the second term, in which only the second integral

$$\int_{-\infty}^{\infty} dx\left(\frac{d\psi}{dx}\right)^2$$

remains. Putting all this together we get

$$\langle x\frac{dV}{dx}\rangle + \langle V\rangle = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\left(\frac{d\psi}{dx}\right)^2 + E\int_{-\infty}^{\infty} dx\psi^2 = \langle\frac{p^2}{2m}\rangle + E$$

so that

$$\frac{1}{2}\langle x\frac{dV}{dx}\rangle = \langle\frac{p^2}{2m}\rangle$$

CHAPTER 5.

1. We are given

$$\int_{-\infty}^{\infty} dx (A\Psi(x))^* \Psi(x) = \int_{-\infty}^{\infty} dx \Psi(x)^* A\Psi(x)$$

Now let $\Psi(x) = \phi(x) + \lambda\psi(x)$, where λ is an arbitrary complex number. Substitution into the above equation yields, on the l.h.s.

$$\int_{-\infty}^{\infty} dx (A\phi(x) + \lambda A\psi(x))^* (\phi(x) + \lambda\psi(x))$$
$$= \int_{-\infty}^{\infty} dx \left[(A\phi)^* \phi + \lambda (A\phi)^* \psi + \lambda^* (A\psi)^* \phi + |\lambda|^2 (A\psi)^* \psi \right]$$

On the r.h.s. we get

$$\int_{-\infty}^{\infty} dx (\phi(x) + \lambda\psi(x))^* (A\phi(x) + \lambda A\psi(x))$$
$$= \int_{-\infty}^{\infty} dx \left[\phi^* A\phi + \lambda^* \psi^* A\phi + \lambda\phi^* A\psi + |\lambda|^2 \psi^* A\psi \right]$$

Because of the hermiticity of A, the first and fourth terms on each side are equal. For the rest, sine λ is an arbitrary complex number, the coefficients of λ and λ^* are independent , and we may therefore identify these on the two sides of the equation. If we consider λ, for example, we get

$$\int_{-\infty}^{\infty} dx (A\phi(x))^* \psi(x) = \int_{-\infty}^{\infty} dx \phi(x)^* A\psi(x)$$

the desired result.

2. We have $A^+ = A$ and $B^+ = B$, therefore $(A + B)^+ = (A + B)$. Let us call $(A + B) = X$. We have shown that X is hermitian. Consider now

$$(X^+)^n = X^+ X^+ X^+ \ldots X^+ = X X X \ldots X = (X)^n$$

which was to be proved.

3. We have

$$\langle A^2 \rangle = \int_{-\infty}^{\infty} dx \psi^* (x) A^2 \psi(x)$$

Now define $A\psi(x) = \phi(x)$. Then the above relation can be rewritten as

$$\langle A^2\rangle = \int_{-\infty}^{\infty} dx\, \psi(x) A\phi(x) = \int_{-\infty}^{\infty} dx (A\psi(x))^* \phi(x)$$
$$= \int_{-\infty}^{\infty} dx (A\psi(x))^* A\psi(x) \geq 0$$

4. Let $U = e^{iH} = \sum_{n=0}^{\infty} \frac{i^n H^n}{n!}$. Then $U^+ = \sum_{n=0}^{\infty} \frac{(-i)^n (H^n)^+}{n!} = \sum_{n=0}^{\infty} \frac{(-i)^n (H^n)}{n!} = e^{-iH}$, and thus

the hermitian conjugate of e^{iH} is e^{-iH} provided $H = H^+$.

5. We need to show that

$$e^{iH} e^{-iH} = \sum_{n=0}^{\infty} \frac{i^n}{n!} H^n \sum_{m=o}^{\infty} \frac{(-i)^m}{m!} H^m = 1$$

Let us pick a particular coefficient in the series, say $k = m + n$ and calculate its coefficient. We get, with $m = k - n$, the coefficient of H^k is

$$\sum_{n=0}^{k} \frac{i^n}{n!} \frac{(-i)^{k-n}}{(k-n)!} = \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} i^n (-i)^{k-n}$$
$$= \frac{1}{k!} (i - i)^k = 0$$

Thus in the product only the $m = n = 0$ term remains, and this is equal to unity.

6. We write $I(\lambda, \lambda^*) = \int_{-\infty}^{\infty} dx \big(\phi(x) + \lambda\psi(x)\big)^* (\phi(x) + \lambda\psi(x)) \geq 0$. The left hand side, in abbreviated notation can be written as

$$I(\lambda, \lambda^*) = \int |\phi|^2 + \lambda^* \int \psi^* \phi + \lambda \int \phi^* \psi + \lambda\lambda^* \int |\psi|^2$$

Since λ and λ^* are independent, he minimum value of this occurs when

$$\frac{\partial I}{\partial \lambda^*} = \int \psi^* \phi + \lambda \int |\psi|^2 = 0$$
$$\frac{\partial I}{\partial \lambda} = \int \phi^* \psi + \lambda^* \int |\psi|^2 = 0$$

When these values of λ and λ^* are inserted in the expression for $I(\lambda,\lambda^*)$ we get

$$I(\lambda_{min}, \lambda^*_{min}) = \int |\phi|^2 - \frac{\int \phi^* \psi \int \psi^* \phi}{\int |\psi|^2} \geq 0$$

from which we get the Schwartz inequality.

7. We have $UU^+ = 1$ and $VV^+ = 1$. Now $(UV)^+ = V^+U^+$ so that

$$(UV)(UV)^+ = UVV^+U^+ = UU^+ = 1$$

8. Let $U\psi(x) = \lambda\psi(x)$, so that λ is an eigenvalue of U. Since U is unitary, $U^+U = 1$. Now

$$\int_{-\infty}^{\infty} dx (U\psi(x))^* U\psi(x) = \int_{-\infty}^{\infty} dx \psi^*(x) U^+ U\psi(x) =$$
$$= \int_{-\infty}^{\infty} dx \psi^*(x)\psi(x) = 1$$

On the other hand, using the eigenvalue equation, the integral may be written in the form

$$\int_{-\infty}^{\infty} dx (U\psi(x))^* U\psi(x) = \lambda^*\lambda \int_{-\infty}^{\infty} dx \psi^*(x)\psi(x) = |\lambda|^2$$

It follows that $|\lambda|^2 = 1$, or equivalently $\lambda = e^{ia}$, with a real.

9. We write

$$\int_{-\infty}^{\infty} dx \phi(x)^* \phi(x) = \int_{-\infty}^{\infty} dx (U\psi(x))^* U\psi(x) = \int_{-\infty}^{\infty} dx \psi^*(x) U^+ U\psi(x) =$$
$$= \int_{-\infty}^{\infty} dx \psi^*(x)\psi(x) = 1$$

10. We write, in abbreviated notation

$$\int v_a^* v_b = \int (Uu_a)^* Uu_b = \int u_a^* U^+ U u_b = \int u_a^* u_b = \delta_{ab}$$

11. (a) We are given $A^+ = A$ and $B^+ = B$. We now calculate

$(i[A,B])^+ = (iAB - iBA)^+ = -i(AB)^+ - (-i)(BA)^+ = -i(B^+A^+) + i(A^+B^+)$

$= -iBA + iAB = i[A,B]$

(b) $[AB,C] = ABC\text{-}CAB = ABC - ACB + ACB - CAB = A(BC - CB) - (AC - CA)B$

$= A[B,C] - [A,C]B$

(c) The Jacobi identity written out in detail is

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$

$A(BC – CB) – (BC – CB)A + B(CA – AC) – (CA - AC)B + C(AB – BA) – (AB – BA)C$

$= ABC – ACB – BCA + CBA + BCA – BAC – CAB + ACB + CAB – CBA – ABC + BAC$

It is easy to see that the sum is zero.

12. We have

$e^A B e^{-A} = (1 + A + A^2/2! + A^3/3! + A^4/4! +\ldots)B (1 - A + A^2/2! - A^3/3! + A^4/4! -\ldots)$

Let us now take the term independent of A: it is B.
The terms of first order in A are $AB – BA = [A,B]$.
The terms of second order in A are

$A^2B/2! – ABA + BA^2/2! = (1/2!)(A^2B – 2ABA + BA^2)$

$= (1/2!)(A(AB – BA) – (AB – BA)A) = (1/2!)\{A[A,B]-[A,B]A\}$

$= (1/2!)[A,[A,B]]$

The terms of third order in A are $A^3B/3! – A^2BA/2! + ABA^2/2! – BA^3$. One can again rearrange these and show that this term is $(1/3!)[A,[A,[A,B]]]$.

There is actually a neater way to do this. Consider

$$F(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

Then

$$\frac{dF(\lambda)}{d\lambda} = e^{\lambda A} ABe^{-\lambda A} - e^{\lambda A} BAe^{-\lambda A} = e^{\lambda A}[A,B]e^{-\lambda A}$$

Differentiating again we get

$$\frac{d^2F(\lambda)}{d\lambda^2} = e^{\lambda A}[A,[A,B]]e^{-\lambda A}$$

and so on. We now use the Taylor expansion to calculate $F(1) = e^A B e^{-A}$.

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \frac{1}{3!}F'''(0) + ..,$$
$$= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ...$$

13. Consider the eigenvalue equation $Hu = \lambda u$. Applying H to this equation we get

$H^2 u = \lambda^2 u$; $H^3 u = \lambda^3 u$ and $H^4 u = \lambda^4 u$. We are given that $H^4 = 1$, which means that H^4 applied to any function yields 1. In particular this means that $\lambda^4 = 1$. The solutions of this are $\lambda = 1, -1, i$, and $-i$. However, H is hermitian, so that the eigenvalues are real. Thus only $\lambda = \pm 1$ are possible eigenvalues. If H is not hermitian, then all four eigenvalues are acceptable.

14. We have the equations

$$Bu_a^{(1)} = b_{11}u_a^{(1)} + b_{12}u_a^{(2)}$$
$$Bu_a^{(2)} = b_{21}u_a^{(1)} + b_{22}u_a^{(2)}$$

Let us now introduce functions $(v_a^{(1)}, v_a^{(2)})$ that satisfy the equations $Bv_a^{(1)} = b_1 v_a^{(1)}; Bv_a^{(2)} = b_2 v_a^{(2)}$. We write, with simplified notation,

$$v_1 = \alpha\, u_1 + \beta\, u_2$$
$$v_2 = \gamma\, u_1 + \delta\, u_2$$

The b_1 - eigenvalue equation reads

$$b_1 v_1 = B\,(\,\alpha\, u_1 + \beta\, u_2) = \alpha\,(b_{11}\, u_1 + b_{12} u_2) + \beta\,(b_{21} u_1 + b_{22} u_2)$$

We write the l.h.s. as $b_1(\alpha\, u_1 + \beta\, u_2)$. We can now take the coefficients of u_1 and u_2 separately, and get the following equations

$$\alpha\,(b_1 - b_{11}) = \beta b_{21}$$
$$\beta\,(b_1 - b_{22}) = \alpha b_{12}$$

The product of the two equations yields a quadratic equation for b_1, whose solution is

$$b_1 = \frac{b_{11} + b_{22}}{2} \pm \sqrt{\frac{(b_{11} - b_{22})^2}{4} + b_{12} b_{21}}$$

We may choose the + sign for the b_1 eigenvalue. An examination of the equation involving v_2 leads to an identical equation, and we associate the – sign with the b_2 eigenvalue. Once we know the eigenvalues, we can find the ratios α/β and γ/δ. These suffice, since the normalization condition implies that

$$\alpha^2 + \beta^2 = 1 \text{ and } \gamma^2 + \delta^2 = 1$$

15. The equations of motion for the expectation values are

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle[H,x]\rangle = \frac{i}{\hbar}\langle[\frac{p^2}{2m},x]\rangle = \frac{i}{m\hbar}\langle p[p,x]\rangle = \langle\frac{p}{m}\rangle$$
$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle[H,p]\rangle = -\frac{i}{\hbar}\langle[p,\frac{1}{2}m\omega_1^2x^2+\omega_2 x]\rangle = -m\omega_1^2\langle x\rangle - \omega_2$$

16. We may combine the above equations to get

$$\frac{d^2}{dt^2}\langle x\rangle = -\omega_1^2\langle x\rangle - \frac{\omega_2}{m}$$

The solution of this equation is obtained by introducing the variable

$$X = \langle x\rangle + \frac{\omega_2}{m\omega_1^2}$$

The equation for X reads $d^2X/dt^2 = -\omega_1^2 X$, whose solution is

$$X = A\cos\omega_1 t + B\sin\omega_1 t$$

This gives us

$$\langle x\rangle_t = -\frac{\omega_2}{m\omega_1^2} + A\cos\omega_1 t + B\sin\omega_1 t$$

At $t = 0$

$$\langle x\rangle_0 = -\frac{\omega_2}{m\omega_1^2} + A$$
$$\langle p\rangle_0 = m\frac{d}{dt}\langle x\rangle_{t=0} = mB\omega_1$$

We can therefore write A and B in terms of the initial values of $< x >$ and $< p >$,

$$\langle x\rangle_t = -\frac{\omega_2}{m\omega_1^2} + \left(\langle x\rangle_0 + \frac{\omega_2}{m\omega_1^2}\right)\cos\omega_1 t + \frac{\langle p\rangle_0}{m\omega_1}\sin\omega_1 t$$

17. We calculate as above, but we can equally well use Eq. (5-53) and (5-57), to get

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$
$$\frac{d}{dt}\langle p\rangle = -\langle\frac{\partial V(x,t)}{\partial x}\rangle = eE_0 cos\omega t$$

Finally

$$\frac{d}{dt}\langle H\rangle = \langle\frac{\partial H}{\partial t}\rangle = eE_0\omega\sin\omega t\langle x\rangle$$

18. We can solve the second of the above equations to get

$$\langle p\rangle_t = \frac{eE_0}{\omega}\sin\omega t + \langle p\rangle_{t=0}$$

This may be inserted into the first equation, and the result is

$$\langle x\rangle_t = -\frac{eE_0}{m\omega^2}(\cos\omega t - 1) + \frac{\langle p\rangle_{t=0}\,t}{m} + \langle x\rangle_{t=0}$$

CHAPTER 6

19. (a) We have

$$A|a> = a|a>$$

It follows that

$$<a|A|a> = a<a|a> = a$$

if the eigenstate of *A* corresponding to the eigenvalue a is normalized to unity. The complex conjugate of this equation is

$$<a|A|a>^* = <a|A^+|a> = a^*$$

If $A^+ = A$, then it follows that $a = a^*$, so that a is real.

13. We have

$$\langle \psi | (AB)^+ | \psi \rangle = \langle (AB)\psi | \psi \rangle = \langle B\psi | A^+ | \psi \rangle = \langle \psi | B^+ A^+ | \psi \rangle$$

This is true for every ψ, so that $(AB)^+ = B^+A^+$

2.

$$\begin{aligned} TrAB &= \sum_n \langle n | AB | n \rangle = \sum_n \langle n | A\mathbf{1}B | n \rangle \\ &= \sum_n \sum_m \langle n | A | m \rangle \langle m | B | n \rangle = \sum_n \sum_m \langle m | B | n \rangle \langle n | A | m \rangle \\ &= \sum_m \langle m | B\mathbf{1}A | m \rangle = \sum_m \langle m | BA | m \rangle = TrBA \end{aligned}$$

3. We start with the definition of $|n>$ as

$$| n \rangle = \frac{1}{\sqrt{n!}} (A^+)^n \, | 0 \rangle$$

We now take Eq. (6-47) from the text to see that

$$A | n \rangle = \frac{1}{\sqrt{n!}} A(A^+)^n \, | 0 \rangle = \frac{n}{\sqrt{n!}} (A^+)^{n-1} \, | 0 \rangle = \frac{\sqrt{n}}{\sqrt{(n-1)!}} (A^+)^{n-1} \, | 0 \rangle = \sqrt{n} \, | n-1 \rangle$$

4. Let $f(A^+) = \sum_{n=1}^{N} C_n (A^+)^n$. We then use Eq. (6-47) to obtain

$$Af(A^+)\,|0\rangle = A\sum_{n=1}^{N} C_n (A^+)^n\,|0\rangle = \sum_{n=1}^{N} nC_n (A^+)^{n-1}\,|0\rangle$$

$$= \frac{d}{dA^+}\sum_{n=1}^{N} C_n (A^+)^n\,|0\rangle = \frac{df(A^+)}{dA^+}\,|0\rangle$$

5. We use the fact that Eq. (6-36) leads to

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}(A^+ - A)$$

We can now calculate

$$\langle k\,|\,x\,|\,n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle k\,|\,A + A^+\,|\,n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle k\,|\,n-1\rangle + \sqrt{k}\langle k-1\,|\,n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}\right)$$

which shows that $k = n \pm 1$.

6. In exactly the same way we show that

$$\langle k\,|\,p\,|\,n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle k\,|\,A^+ - A\,|\,n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n+1}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1})$$

7. Let us now calculate

$$\langle k\,|\,px\,|\,n\rangle = \langle k\,|\,p\mathbf{1}x\,|\,n\rangle = \sum_q \langle k\,|\,p\,|\,q\rangle\langle q\,|\,x\,|\,n\rangle$$

We may now use the results of problems 5 and 6. We get for the above

$$\frac{i\hbar}{2}\sum_q(\sqrt{k}\delta_{k-1,q} - \sqrt{k+1}\delta_{k+1,q})(\sqrt{n}\delta_{q,n-1} + \sqrt{n+1}\delta_{q,n+1})$$

$$= \frac{i\hbar}{2}(\sqrt{kn}\delta_{kn} - \sqrt{(k+1)n}\delta_{k+1,n-1} + \sqrt{k(n+1)}\delta_{k-1,n+1} - \sqrt{(k+1)(n+1)}\delta_{k+1,n+1})$$

$$= \frac{i\hbar}{2}(-\delta_{kn} - \sqrt{(k+1)(k+2)}\delta_{k+2,n} + \sqrt{n(n+2)}\delta_{k,n+2})$$

To calculate $\langle k\,|\,xp\,|\,n\rangle$ we may proceed in exactly the same way. It is also possible to abbreviate the calculation by noting that since x and p are hermitian operators, it follows that

$$\langle k\,|\,xp\,|\,n\rangle = \langle n\,|\,px\,|\,k\rangle^*$$

so that the desired quantity is obtained from what we obtained before by interchanging k and n and complex-conjugating. The latter only changes the overall sign, so that we get

$$\langle k\,|\,xp\,|\,n\rangle = -\frac{i\hbar}{2}(-\delta_{kn} - \sqrt{(n+1)(n+2)}\delta_{k,n+2} + \sqrt{(k+1)(k+2)}\delta_{k+2,n})$$

8.The results of problem 7 immediately lead to

$$\langle k\,|\,xp - px\,|\,n\rangle = i\hbar\delta_{kn}$$

9. This follows immediately from problems 5 and 6.

10. We again use

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$
$$p = i\sqrt{\frac{m\omega\hbar}{2}}(A^+ - A)$$

to obtain the operator expression for

$$x^2 = \frac{\hbar}{2m\omega}(A + A^+)(A + A^+) = \frac{\hbar}{2m\omega}(A^2 + 2A^+A + (A^+)^2 + 1)$$
$$p^2 = -\frac{m\omega\hbar}{2}(A^+ - A)(A^+ - A) = -\frac{m\omega\hbar}{2}(A^2 - 2A^+A + (A^+)^2 - 1)$$

where we have used $[A,A^+] = 1$.

The quadratic terms change the values of the eigenvalue integer by 2, so that they do not appear in the desired expressions. We get, very simply

$$\langle n\,|\,x^2\,|\,n\rangle = \frac{\hbar}{2m\omega}(2n+1)$$
$$\langle n\,|\,p^2\,|\,n\rangle = \frac{m\omega\hbar}{2}(2n+1)$$

14. Given the results of problem 9, and of 10, we have

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}(2n+1)$$
$$(\Delta p)^2 = \frac{\hbar m\omega}{2}(2n+1)$$

and therefore

$$\Delta x \Delta p = \hbar(n+\frac{1}{2})$$

15. The eigenstate in $A|\alpha\rangle = \alpha|\alpha\rangle$ may be written in the form

$$|\alpha\rangle = f(A^+)|0\rangle$$

It follows from the result of problem 4 that the eigenvalue equation reads

$$Af(A^+)|0\rangle = \frac{df(A^+)}{dA^+}|0\rangle = \alpha f(A^+)|0\rangle$$

The solution of $df(x) = \alpha f(x)$ is $f(x) = C e^{\alpha x}$ so that

$$|\alpha\rangle = Ce^{\alpha A^+}|0\rangle$$

The constant C is determined by the normalization condition $\langle\alpha|\alpha\rangle = 1$
This means that

$$\frac{1}{C^2} = \langle 0|e^{\alpha^* A}e^{\alpha A^+}|0\rangle = \sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{n!}\langle 0|\left(\frac{d}{dA^+}\right)^n e^{\alpha A^+}|0\rangle$$
$$= \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\langle 0|e^{\alpha A^+}|0\rangle = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = e^{|\alpha|^2}$$

Consequently

$$C = e^{-|\alpha|^2/2}$$

We may now expand the state as follows

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle = \sum_n |n\rangle\langle 0|\frac{A^n}{\sqrt{n!}}Ce^{\alpha A^+}|0\rangle$$
$$= C\sum_n |n\rangle\frac{1}{\sqrt{n!}}\langle 0|\left(\frac{d}{dA^+}\right)^n e^{\alpha A^+}|0\rangle = C\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

The probability that the state $|\alpha\rangle$ contains n quanta is

$$P_n = |\langle n|\alpha\rangle|^2 = C^2 \frac{|\alpha|^{2n}}{n!} = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$

This is known as the *Poisson distribution.*

Finally

$$\langle\alpha|N|\alpha\rangle = \langle\alpha|A^+A|\alpha\rangle = \alpha^*\alpha = |\alpha|^2$$

13. The equations of motion read

$$\frac{dx(t)}{dt} = \frac{i}{\hbar}[H, x(t)] = \frac{i}{\hbar}[\frac{p^2(t)}{2m}, x(t)] = \frac{p(t)}{m}$$

$$\frac{dp(t)}{dt} = \frac{i}{\hbar}[mgx(t), p(t)] = -mg$$

This leads to the equation

$$\frac{d^2x(t)}{dt^2} = -g$$

The general solution is

$$x(t) = \frac{1}{2}gt^2 + \frac{p(0)}{m}t + x(0)$$

14. We have, as always

$$\frac{dx}{dt} = \frac{p}{m}$$

Also

$$\begin{aligned}\frac{dp}{dt} &= \frac{i}{\hbar}[\frac{1}{2}m\omega^2x^2 + e\xi x, p] \\ &= \frac{i}{\hbar}\left(\frac{1}{2}m\omega^2x[x,p] + \frac{1}{2}m\omega^2[x,p]x + e\xi[x,p]\right) \\ &= -m\omega^2x - e\xi\end{aligned}$$

Differentiating the first equation with respect to t and rearranging leads to

$$\frac{d^2x}{dt^2} = -\omega^2x - \frac{e\xi}{m} = -\omega^2(x + \frac{e\xi}{m\omega^2})$$

The solution of this equation is

$$x + \frac{e\xi}{m\omega^2} = A\cos\omega t + B\sin\omega t$$
$$= \left(x(0) + \frac{e\xi}{m\omega^2}\right)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

We can now calculate the commutator $[x(t_1), x(t_2)]$, which should vanish when $t_1 = t_2$. In this calculation it is only the commutator $[p(0), x(0)]$ that plays a role. We have

$$[x(t_1), x(t_2)] = [x(0)\cos\omega t_1 + \frac{p(0)}{m\omega}\sin\omega t_1, x(0)\cos\omega t_2 + \frac{p(0)}{m\omega}\sin\omega t_2]$$
$$= i\hbar\left(\frac{1}{m\omega}(\cos\omega t_1 \sin\omega t_2 - \sin\omega t_1 \cos\omega t_2)\right) = \frac{i\hbar}{m\omega}\sin\omega(t_2 - t_1)$$

16. We simplify the algebra by writing

$$\sqrt{\frac{m\omega}{2\hbar}} = a; \quad \sqrt{\frac{\hbar}{2m\omega}} = \frac{1}{2a}$$

Then

$$\sqrt{n!}\left(\frac{\hbar\pi}{m\omega}\right)^{1/4} u_n(x) = v_n(x) = \left(ax - \frac{1}{2a}\frac{d}{dx}\right)^n e^{-a^2x^2}$$

Now with the notation $y = ax$ we get

$$v_1(y) = (y - \frac{1}{2}\frac{d}{dy})e^{-y^2} = (y + y)e^{-y^2} = 2ye^{-y^2}$$
$$v_2(y) = (y - \frac{1}{2}\frac{d}{dy})(2ye^{-y^2}) = (2y^2 - 1 + 2y^2)e^{-y^2}$$
$$= (4y^2 - 1)e^{-y^2}$$

Next

$$v_3(y) = (y - \frac{1}{2}\frac{d}{dy})\left[(4y^2 - 1)e^{-y^2}\right]$$
$$= \left(4y^3 - y - 4y + y(4y^2 - 1)\right)e^{-y^2}$$
$$= (8y^3 - 6y)e^{-y^2}$$

The rest is substitution $y = \sqrt{\frac{m\omega}{2\hbar}}x$

17. We learned in problem 4 that

$$Af(A^+)\,|0\rangle = \frac{df(A^+)}{dA^+}\,|0\rangle$$

Iteration of this leads to

$$A^n f(A^+)\,|0\rangle = \frac{d^n f(A^+)}{dA^{+n}}\,|0\rangle$$

We use this to get

$$e^{\lambda A} f(A^+)|0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}A^n f(A^+)\,|0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\frac{d}{dA^+}\right)^n f(A^+)|0\rangle = f(A^+ + \lambda)|0\rangle$$

18. We use the result of problem **16** to write

$$e^{\lambda A} f(A^+)e^{-\lambda A}g(A^+)|0\rangle = e^{\lambda A} f(A^+)g(A^+ - \lambda)\,|0\rangle = f(A^+ + \lambda)g(A^+)\,|0\rangle$$

Since this is true for *any* state of the form $g(A^+)|0>$ we have

$$e^{\lambda A} f(A^+)e^{-\lambda A} = f(A^+ + \lambda)$$

In the above we used the first formula in the solution to **16**, which depended on the fact that $[A,A^+] = 1$. More generally we have the Baker-Hausdorff form, which we derive as follows:

Define

$$F(\lambda) = e^{\lambda A}A^+e^{-\lambda A}$$

Differentiation w.r.t. λ yields

$$\frac{dF(\lambda)}{d\lambda} = e^{\lambda A}AA^+e^{-\lambda A} - e^{\lambda A}A^+Ae^{-\lambda A} = e^{\lambda A}[A,A^+]e^{-\lambda A} \equiv e^{\lambda A}C_1e^{-\lambda A}$$

Iteration leads to

$$\frac{d^2F(\lambda)}{d\lambda^2} = e^{\lambda A}[A,[A,A^+]]e^{-\lambda A} \equiv e^{\lambda A}C_2 e^{-\lambda A}$$

....

$$\frac{d^nF(\lambda)}{d\lambda^n} = e^{\lambda A}[A,[A,[A,[A,....]]..]e^{-\lambda A} \equiv e^{\lambda A}C_n e^{-\lambda A}$$

with A appearing n times in C_n. We may now use a Taylor expansion for

$$F(\lambda+\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}\frac{d^nF(\lambda)}{d\lambda^n} = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}e^{\lambda A}C_n e^{-\lambda A}$$

If we now set $\lambda = 0$ we get

$$F(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}C_n$$

which translates into

$$e^{\sigma A}A^+e^{-\sigma A} = A^+ + \sigma[A,A^+] + \frac{\sigma^2}{2!}[A,[A,A^+]] + \frac{\sigma^3}{3!}[A,[A,[A,A^+]]] + ...$$

Note that if $[A,A^+] = 1$ only the first two terms appear, so that

$$e^{\sigma A}f(A^+)e^{-\sigma A} = f(A^+ + \sigma[A,A^+]) = f(A^+ + \sigma)$$

19. We follow the procedure outlined in the hint. We define $F(\lambda)$ by

$$e^{\lambda(aA+bA^+)} = e^{\lambda aA}F(\lambda)$$

Differentiation w.r.t λ yields

$$(aA+bA^+)e^{\lambda aA}F(\lambda) = aAe^{\lambda A}F(\lambda) + e^{\lambda aA}\frac{dF(\lambda)}{d\lambda}$$

The first terms on each side cancel, and multiplication by $e^{-\lambda aA}$ on the left yields

$$\frac{dF(\lambda)}{d\lambda} = e^{-\lambda aA}bA^+e^{\lambda aA}F(\lambda) = bA^+ - \lambda ab[A,A^+]F(\lambda)$$

When $[A,A^+]$ commutes with A. We can now integrate w.r.t. λ and after integration Set $\lambda = 1$. We then get

$$F(1) = e^{bA^+ - ab[A,A^+]/2} = e^{bA^+}e^{-ab/2}$$

so that

$$e^{aA+bA^+} = e^{aA}e^{bA^+}e^{-ab/2}$$

20. We can use the procedure of problem **17**, but a simpler way is to take the hermitian conjugate of the result. For a *real* function f and λ real, this reads

$$e^{-\lambda A^+} f(A)e^{\lambda A^+} = f(A+\lambda)$$

Changing λ to -λ yields

$$e^{\lambda A^+} f(A)e^{-\lambda A^+} = f(A-\lambda)$$

The remaining steps that lead to

$$e^{aA+bA^+} = e^{bA^+}e^{aA}\ e^{ab/2}$$

are identical to the ones used in problem **18**.

20. For the harmonic oscillator problem we have

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A+A^+)$$

This means that e^{ikx} is of the form given in problem **19** with $a = b = ik\sqrt{\hbar/2m\omega}$

This leads to

$$e^{ikx} = e^{ik\sqrt{\hbar/2m\omega}A^+}e^{ik\sqrt{\hbar/2m\omega}A}e^{-\hbar k^2/4m\omega}$$

Since $A|0\rangle = 0$ and $\langle 0|A^+ = 0$, we get

$$\langle 0|e^{ikx}|0\rangle = e^{-\hbar k^2/4m\omega}$$

21. An alternative calculation, given that $u_0(x) = (m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}$, is

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx e^{ikx}e^{-m\omega x^2\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx e^{-\frac{m\omega}{\hbar}(x-\frac{ik\hbar}{2m\omega})^2}e^{-\frac{\hbar k^2}{4m\omega}}$$

The integral is a simple gaussian integral and $\int_{-\infty}^{\infty} dy e^{-m\omega y^2/\hbar} = \sqrt{\frac{\hbar\pi}{m\omega}}$ which just cancels the factor in front. Thus the two results agree.