Physics 7502 Quantum Mechanics Spring 2019

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Midterm 2

Given: Friday, March 29, 2019 *Time: 55 minutes* *Total marks: 30*

Problem 1: (15 points) A particle of mass m and charge q moves in the $x-y$ plane in the uniform magnetic field $\vec{B} = B\hat{z}$. The sample has length L_1 in the x direction and length L_2 in the y direction. You can assume periodic boundary conditions in each direction.

(i) Write down the Hamiltonian describing the particle, using the gauge $A_x = -By$. (5 points)

(ii) Using a suitable ansatz for the wavefunction, show that the Schrodinger equation for this system can be reduced to a harmonic oscillator, and find the frequency of this harmonic oscillator. (5 points)

(iii) Derive the number of states in each Landau level. (5 points).

Problem 2: (15 points) A harmonic oscillator has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2 \tag{1}$$

with

$$\omega^2(t) = \omega_0^2 + \lambda e^{-\frac{t^2}{\tau^2}} \tag{2}$$

and λ small. The oscillator starts in the ground state $|0\rangle$ at $t \to -\infty$. What is the probability that it is in the second excited state $|2\rangle$ at $t \to \infty$?

[You can use that the creation and annihilation operators are given by

$$A = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}p \tag{3}$$

$$A^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{1}{\sqrt{2m\omega\hbar}}p \tag{4}$$

and the integral of a gaussian is

$$\int_{z=-\infty}^{\infty} dz e^{-az^2+bz} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \qquad] \tag{5}$$

Solution: (1) (i) In the gauge

$$\vec{A} = -By\hat{x} \tag{6}$$

we have

$$A_x = -By, \quad A_y = 0, \quad A_z = 0 \tag{7}$$

Thus we have

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 = \frac{1}{2m}((p_x + qBy)^2 + p_y^2) = \frac{1}{2m}(p_x^2 + 2qBp_xy + q^2B^2y^2 + p_y^2) \tag{8}$$

(ii) We take

$$\psi = e^{ikx}\psi(y) \tag{9}$$

$$H\psi = \frac{1}{2m}(\hbar^2k^2 + 2qB\hbar ky + q^2B^2y^2 + p_y^2) = \frac{1}{2m}(p_y^2 + q^2B^2(y + \frac{\hbar k}{qB})^2) \tag{10}$$

This is a harmonic oscillator, with

$$\frac{1}{2}m\omega^2 = \frac{1}{2m}q^2B^2, \quad \omega = \frac{eB}{m} \tag{11}$$

Thus the energy levels are

$$(n + \frac{1}{2})\hbar\omega \tag{12}$$

Note that k is arbitrary, so the levels are highly degenerate.

(iii) We have a strip of length L_1 in the x direction. Then

$$k = \frac{2\pi n}{L_1} \tag{13}$$

Thus the y locations, for a given value of the excitation of the harmonic oscillator, are

$$y_n = -\frac{\hbar k}{qB} = -\frac{2\pi\hbar n^*}{L_1qB} \tag{14}$$

This has the range $0 \le y \le L_2$. Thus

$$\frac{2\pi\hbar|n^*_{max}|}{L_1qB} = L_2 \tag{15}$$

which gives

$$|n^*_{max}| = \frac{qBL_1L_2}{2\pi\hbar} = \frac{qBL_1L_2}{h} \tag{16}$$

This is the number of states per Landau level.

Solution: (2) We have

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 + \frac{1}{2}m\lambda q(t)\hat{x}^2 \tag{17}$$

where

$$q(t) = e^{-\frac{t^2}{\tau^2}} \tag{18}$$

Thus

$$V(t) = \frac{1}{2}mq(t)\hat{x}^2 \tag{19}$$

We have

$$|\phi_k\rangle = \frac{1}{\sqrt{k!}}(\hat{A}^\dagger)^k|0\rangle \tag{20}$$

We also have

$$A = \sqrt{\frac{m\omega_0}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega_0\hbar}}p \tag{21}$$

$$A^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}x - i\frac{1}{\sqrt{2m\omega_0\hbar}}p \tag{22}$$

Thus

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{A} + \hat{A}^\dagger) \tag{23}$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega_0}(\hat{A} + \hat{A}^\dagger)^2 \tag{24}$$

Thus

$$V(t) = \frac{1}{2}mq(t)\frac{\hbar}{2m\omega_0}(\hat{A} + \hat{A}^\dagger)^2 = q(t)\frac{\hbar}{4\omega_0}(\hat{A} + \hat{A}^\dagger)^2 \tag{25}$$

Thus we get

$$\langle\phi_l|V(t)|\phi_k\rangle = q(t)\frac{\hbar}{4\omega_0}\frac{1}{\sqrt{l!}}\frac{1}{\sqrt{k!}}\langle 0|\hat{A}^l(\hat{A} + \hat{A}^\dagger)^2(\hat{A}^\dagger)^k|0\rangle \tag{26}$$

We set

$$|\phi_k\rangle = |0\rangle, \quad \langle\phi_l| = \langle 2| \tag{27}$$

and compute

$$\langle\phi_l|V(t)|\phi_k\rangle = q(t)\frac{\hbar}{4\omega_0}\frac{1}{\sqrt{2}}\langle 0|\hat{A}^2(\hat{A} + \hat{A}^\dagger)^2|0\rangle \tag{28}$$

$$= q(t)\frac{\hbar}{4\omega_0}\frac{1}{\sqrt{2}}\langle 0|\hat{A}^2(\hat{A}^\dagger)^2|0\rangle = 2q(t)\frac{\hbar}{4\omega_0}\frac{1}{\sqrt{2}} = q(t)\frac{\hbar}{2\omega_0}\frac{1}{\sqrt{2}} \tag{29}$$

We have

$$c_l(t) = c_l(0) - \frac{i}{\hbar}\lambda \sum_k c_k(0) \int_{t'=-\infty}^{t} dt' e^{-i\frac{(E_k - E_l)}{\hbar}t'} \langle \phi_l | V(t') | \phi_k \rangle \tag{30}$$

Thus

$$c_2(t) = -\frac{i}{\hbar}\lambda \int_{t'=-\infty}^{t} dt' e^{i2\omega_0 t'} e^{-\frac{t'^2}{\tau^2}} \frac{\hbar}{2\omega_0} \frac{1}{\sqrt{2}} \tag{31}$$

$$c_2(\infty) = -\frac{i}{2\omega_0}\frac{1}{\sqrt{2}}\lambda \int_{t'=-\infty}^{\infty} dt' e^{i2\omega_0 t'} e^{-\frac{t'^2}{\tau^2}} = -\frac{i}{2\omega_0}\frac{1}{\sqrt{2}}\lambda\sqrt{\pi}\tau e^{-\frac{4\omega_0^2\tau^2}{4}} \tag{32}$$

$$= -\frac{i}{2\omega_0}\frac{1}{\sqrt{2}}\lambda\sqrt{\pi}\tau e^{-\omega_0^2\tau^2} \tag{33}$$

The probability is

$$P_2 = |c_2(\infty)|^2 = \frac{\pi\lambda^2\tau^2}{8\omega_0^2} e^{-2\omega_0^2\tau^2} \tag{34}$$