Number (NOT NAME): ____________________

Physics 7502 Quantum Mechanics Spring 2019

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Midterm 1

Given: Friday, Feb 22, 2019 *Time: 55 minutes* *Total marks: 45*

Problem 1: (15 points) A particle has spin $s = \frac{1}{2}$, and an orbital angular momentum $l = 1$. The total angular momentum is defined by

$$\vec{J} = \vec{L} + \vec{S} \tag{1}$$

(i) What are the allowed values of the quantum number j? You can just give the allowed values of j; you do not have to derive them. (Here j is defined by $J^2|\psi\rangle = \hbar^2 j(j+1)|\psi\rangle$.) (5 points)

(ii) Suppose the Hamiltonian is

$$\hat{H} = \alpha J^2 - \beta \vec{L} \cdot \vec{S} \tag{2}$$

Find the eigenvalues of $\hat{H}$. (10 points)

Problem 2: (15 points) Two particles (called particle 1 and particle 2), each with spin $\frac{1}{2}$, are in a singlet state. What is the probability that the states of the two particles are found to be

$$|\psi\rangle_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \quad |\psi\rangle_2 = \begin{pmatrix} -\frac{4i}{5} \\ \frac{3i}{5} \end{pmatrix} \tag{3}$$

Problem 3: (15 points) Consider a 1-dimensional infinite square well of width a. We place N noninteracting electrons in this well, in the configuration with the lowest allowed energy. Assume that N is even and that each electron has mass m. Find the pressure on each wall.

Solution: (1) (a) We have

$$j = \frac{3}{2}, \quad j = \frac{1}{2} \tag{4}$$

(b) We have

$$\vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) \tag{5}$$

$$\hat{H} = \alpha\hbar^2 j(j+1) - \frac{1}{2}\beta(\hbar^2(j(j+1) - \hbar^2 l(l+1) - \hbar^2 s(s+1)) \tag{6}$$

Thus the energies are

$$j = \frac{3}{2}: \quad E = \hbar^2(\alpha\frac{3}{2}\frac{5}{2} - \frac{1}{2}\beta(\frac{3}{2}\frac{5}{2} - 2 - \frac{3}{4}) = \hbar^2(\frac{15}{4}\alpha - \frac{1}{2}\beta) \tag{7}$$

$$j = \frac{1}{2}: \quad E = \hbar^2(\alpha\frac{1}{2}\frac{3}{2} - \frac{1}{2}\beta(\frac{1}{2}\frac{3}{2} - 2 - \frac{3}{4}) = \hbar^2(\frac{3}{4}\alpha + \beta) \tag{8}$$

Solution: (2) We have

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 \tag{9}$$

We take the inner product with

$$|\chi\rangle = \begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}\begin{pmatrix}-\frac{4i}{5}\\\frac{3i}{5}\end{pmatrix} \tag{10}$$

We find

$$\langle\chi|\psi\rangle = \frac{1}{\sqrt{2}}\frac{3}{5}(-\frac{3i}{5}) - \frac{1}{\sqrt{2}}\frac{4}{5}(\frac{4i}{5}) = -\frac{9i}{25\sqrt{2}} - \frac{16i}{25\sqrt{2}} = -\frac{i}{\sqrt{2}} \tag{11}$$

Thus the probability is

$$P = |\langle\chi|\psi\rangle|^2 = \frac{1}{2} \tag{12}$$

Solution: (3) Let the well be $0 < x < a$. The energy eigenstates are

$$|\psi_n\rangle = A_n\sin(k_n x), \quad k_n a = n\pi, \quad k = \frac{n\pi}{a}, \quad n = 1, 2, \ldots \tag{13}$$

Thus

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2} \tag{14}$$

There are two electrons in each level. Thus we fill all levels to $\frac{N}{2}$. Thus the total energy is

$$E_T = 2\times\frac{\hbar^2\pi^2}{2ma^2}\sum_{n=1}^{\frac{N}{2}} n^2 = \frac{\hbar^2\pi^2}{ma^2}\frac{1}{6}\frac{N}{2}(\frac{N}{2}+1)(N+1) \tag{15}$$

The pressure is

$$P = -\frac{\partial E_T}{\partial a} = \frac{\hbar^2\pi^2}{3ma^3}\frac{N}{2}(\frac{N}{2}+1)(N+1) \tag{16}$$